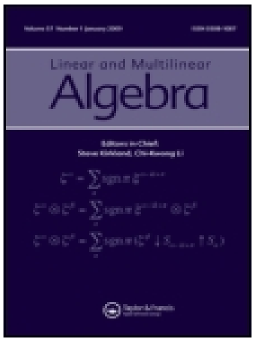

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# On zero product determined algebras

Daniel Brice and Huajun Huang*

*Department of Mathematics and Statistics, Auburn University, Auburn, AL, USA*

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Let $K$ be a commutative ring with identity. A $K$-algebra $A$ is said to be *zero product determined* if for every $K$-bilinear $\varphi$ having the property that $\varphi(a_1, a_2) = 0$ whenever $a_1 a_2 = 0$, there is a $K$-linear $\tilde{\varphi} : A^2 \longrightarrow \operatorname{Im} \varphi$ such that $\varphi(a_1, a_2) = \tilde{\varphi}(a_1 a_2)$ for all $a_1, a_2 \in A$. We provide a necessary and sufficient condition for an algebra $A$ to be zero product determined and use the condition to derive several new results. Among these, we show that the direct sum of algebras is zero product determined if and only if each component algebra is zero product determined; we show that the tensor product of zero product determined algebras is zero product determined in case $K$ is a field or in case the algebra multiplications are surjective; we produce conditions under which the homomorphic images of a zero product determined algebra are zero product determined; finally, we introduce a class of zero product determined matrix algebras that generalizes block upper triangular matrices and extends a result of Brešar, Grašič, and Ortega in 2009.

**Keywords:** zero product determined; algebra homomorphism; tensor product; ladder matrix

**AMS Subject Classifications:** 47B49 (16S50, 15A04, 16W10)

## 1. Introduction

Let $K$ be a commutative ring with identity 1. Given a $K$-algebra $A$ and a $K$-bilinear map $\varphi : A \times A \longrightarrow B$, we may ask whether or not $\varphi$ may be written as the composition of multiplication in $A$ with a $K$-linear map $\tilde{\varphi}$: that is, whether or not

$$\varphi(a_1, a_2) = \tilde{\varphi}(a_1 a_2), \quad \forall a_1, a_2 \in A$$

for some $\tilde{\varphi} : A^2 \longrightarrow B$. (Here and throughout, $A^2$ denotes the $K$-linear span of the products of members of $A$).

In order to study the above problem, Brešar, Grašič, and Sánchez Ortega introduced the notion of a zero product determined algebra in [1]. A $K$-algebra $A$ (not necessarily associative) is called *zero product determined* if each $K$-bilinear map $\varphi : A \times A \longrightarrow B$ satisfying

$$\varphi(a_1, a_2) = 0 \ \text{ whenever } a_1 a_2 = 0$$

can be written as $\varphi(a_1, a_2) = \tilde{\varphi}(a_1 a_2)$ for some $\tilde{\varphi} : A^2 \longrightarrow B$. Their definition was motivated by applications to the study of zero product preserving linear maps defined on

*Corresponding author. Email: huanghu@auburn.edu

Banach algebras and on matrix algebras under the standard matrix product, the Lie product and the Jordan product.[1–5] To illustrate, let $A$, $B$ be $K$-algebras, and let $f : A \longrightarrow B$ be $K$-linear. $f$ is said to be zero product preserving if $f(a_1)f(a_2) = 0$ whenever $a_1 a_2 = 0$. One would like to find conditions on $f$ or on $A$ that imply that $f$ is a homomorphism of $K$-algebras, or is at least close to an algebra homomorphism in some sense. We define a mapping $\varphi(a_1, a_2) = f(a_1)f(a_2)$. Then $\varphi$ is $K$-bilinear and satisfies $\varphi(a_1, a_2) = 0$ whenever $a_1 a_2 = 0$. If $A$ is known to be zero product determined, then there is a unique $K$-linear $\tilde{\varphi} : A^2 \longrightarrow B$ satisfying

$$\tilde{\varphi}(a_1 a_2) = \varphi(a_1, a_2) = f(a_1)f(a_2), \quad \forall a_1, a_2 \in A.$$

If we further assume that $1_A \in A$, then

$$\tilde{\varphi}(a) = \tilde{\varphi}(1_A a) = \varphi(1_A, a) = f(1_A)f(a), \quad \forall a \in A,$$

and by combining the above equations, we arrive at

$$f(a_1)f(a_2) = f(1_A)f(a_1 a_2), \quad \forall a_1, a_2 \in A.$$

As a corollary, if $A$ is zero product determined, and if $1_A \in A$ and $1_B \in B$ are identities, then any zero product preserving linear map $f : A \longrightarrow B$ that satisfies $f(1_A) = 1_B$ is an algebra homomorphism.

The initial work of Brešar, Grašič, and Sánchez Ortega and subsequent work by Ge, Grašič, Li, and Wang have provided examples of zero product determined algebras and algebras that are not zero product determined.[5,6] Recent similar work includes Brešar's and Šmerl's study of commutativity preserving linear maps [7] and Chen's, Wang's, and Yu's study of idempotent preserving bilinear maps and the notion of an idempotent elements determined algebra.[8]

In this paper, we provide several main results that compliment the existing body of research. We reformulate the definition of a zero product determined algebra in terms of tensor products and obtain a necessary and sufficient condition for an algebra to be zero product determined (Theorem 2.3). This reformulation allows us to prove that the direct sum of algebras $\bigoplus_{i \in I} A_i$ for any index set $I$ is zero product determined if and only if each component algebras $A_i$ is zero product determined (Theorem 3.1), and that the tensor product of zero product determined algebras is zero product determined when $K$ is a field or the algebra multiplications are surjective (Theorem 4.3). We go on to examine homomorphic images of zero product determined algebras (Theorem 5.2, Corollaries 5.3 and 5.4). Finally, we define a class of matrix algebras that generalize block upper triangular matrices (Definition 6.1) and provide conditions under which algebras comprised of these matrices are zero product determined (Theorems 6.6 and 6.7), extending the result of [1, Theorem 2.1] that the matrix algebra $M_n(A)$ for a unital $K$-algebra $A$ and $n \geq 2$ is zero product determined.

## 2. Algebras in terms of tensor products

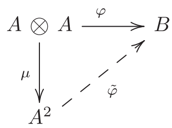
*Definition 2.1* A *$K$-algebra* is a pair $(A, \mu)$ where $A$ is a $K$-(bi)module and $\mu : A \otimes_K A \longrightarrow A$ is a $K$-linear map (i.e. a $K$-module homomorphism).

This definition encompasses associative algebras, alternative algebras (an example being the octonions), Leibniz algebras, Lie algebras, and Jordan algebras, among others. The algebra multiplication is encoded by the map $\mu$ if we define $a_1 a_2 = \mu(a_1 \otimes a_2)$. In this context, $A^2$ (the $K$-linear span of products of members of $A$) is seen to coincide with $\operatorname{Im}\mu$. We will suppress references to the scalar ring $K$ when there is no danger of ambiguity.

*Definition 2.2* The algebra $(A, \mu)$ is called *zero product determined* if each $K$-linear map $\varphi : A \otimes A \longrightarrow B$ satisfying

$$\varphi(a_1 \otimes a_2) = 0 \quad \text{whenever} \quad \mu(a_1 \otimes a_2) = 0$$

factors through $\mu$ as $\varphi = \tilde{\varphi} \circ \mu$ for some $K$-linear map $\tilde{\varphi} : A^2 \longrightarrow B$.




This definition is in agreement with that given in [1]. We next define

$$T_\mu = \{a_1 \otimes a_2 \in A \otimes A \,|\, \mu(a_1 \otimes a_2) = 0\}.$$

In other words, $T_\mu$ consists of elementary tensors in $\operatorname{Ker}\mu$. We use $\langle T_\mu \rangle$ to denote the submodule of $A$ generated by $T_\mu$.

Our first main result, stated below, provides a necessary and sufficient condition for an algebra $(A, \mu)$ to be zero product determined. We will return to this result throughout the sequel.

Theorem 2.3 *$(A, \mu)$ is zero product determined if and only if*

$$\langle T_\mu \rangle = \operatorname{Ker}\mu.$$

*Proof* Let $\varphi : A \otimes A \longrightarrow B$ be an arbitrary liner map satisfying $\varphi(a_1 \otimes a_2) = 0$ whenever $\mu(a_1 \otimes a_2) = 0$. Equivalently, $a_1 \otimes a_2 \in T_\mu$ implies $a_1 \otimes a_2 \in \operatorname{Ker}\varphi$, so $\langle T_\mu \rangle \subseteq \operatorname{Ker}\varphi$.

Now, if $\langle T_\mu \rangle = \operatorname{Ker}\mu$, the above inclusion becomes $\operatorname{Ker}\mu \subseteq \operatorname{Ker}\varphi$, so $\varphi$ factors through $\mu$. Therefore, $(A, \mu)$ is zero product determined.

On the other hand, suppose $\langle T_\mu \rangle \subsetneq \operatorname{Ker}\mu$. Let $P : A \otimes A \longrightarrow A \otimes A / \langle T_\mu \rangle$ be the canonical projection $P : x \longmapsto x + \langle T_\mu \rangle$. $P$ satisfies the requirement that $P(a_1 \otimes a_2) = 0$ whenever $\mu(a_1 \otimes a_2) = 0$, but since $\operatorname{Ker}\mu \nsubseteq \operatorname{Ker} P = \langle T_\mu \rangle$, $P$ does not factor through $\mu$, so $(A, \mu)$ is not zero product determined. □

The theorem establishes that an algebra $A$ is zero product determined if and only if the kernel of its multiplication map is generated by elementary tensors. We illustrate the use of the theorem by providing two new negative examples.

Proposition 2.4 *Let $V$ be a vector space with* $\dim V \geq 2$ *over a field $K$. Then the tensor algebra $\mathcal{T}(V)$ and the symmetric algebra $\mathcal{S}(V)$ are not zero product determined.*

*Proof* Recall that the tensor algebra $\mathcal{T}(V)$ over a vector space $V$ may be thought of as the free unital associative $K$-algebra on dim $V$ generators. Likewise, the symmetric algebra $\mathcal{S}(V)$ over $V$ may be thought of as the free commutative unital associative $K$-algebra on dim $V$ generators. We will denote multiplication in either algebra by juxtaposition to avoid confusion with the member of $\mathcal{T}(V) \otimes \mathcal{T}(V)$ or $\mathcal{S}(V) \otimes \mathcal{S}(V)$.

The tensor and symmetric algebras over a vector space are integral domains, and as such, we have that $\mu(t_1 \otimes t_2) = 0$ if and only if $t_1 = 0$ or $t_2 = 0$, in either case giving $t_1 \otimes t_2 = 0$. In short, this means $T_\mu = 0$. To show that these algebras are not zero product determined, we must now show that Ker $\mu \neq 0$.

Since dim $V \geq 2$, we may select two linearly independent vectors $v_1, v_2 \in V$. Then for the tensor algebra $\mathcal{T}(V)$, consider the element

$$v_1 v_2 \otimes v_1 - v_1 \otimes v_2 v_1 \in \mathcal{T}(V) \otimes \mathcal{T}(V).$$

We have that

$$\mu(v_1 v_2 \otimes v_1 - v_1 \otimes v_2 v_1) = (v_1 v_2) v_1 - v_1 (v_2 v_1) = 0$$

so that $v_1 v_2 \otimes v_1 - v_1 \otimes v_2 v_1 \in \operatorname{Ker} \mu$, while also $v_1 v_2 \otimes v_1 - v_1 \otimes v_2 v_1 \neq 0$ by the linear independence of each of $v_1$, $v_2$, $v_1 v_2$, and $v_2 v_1$.

As for the symmetric algebra $\mathcal{S}(V)$, we note that

$$v_1 \otimes v_2 - v_2 \otimes v_1 \in \mathcal{S}(V) \otimes \mathcal{S}(V)$$

is non-zero by linear independence yet contained in Ker $\mu$ by commutativity. □

## 3. Direct sums of algebras

We will now use Theorem 2.3 to study the relationship between zero product determined algebras and direct sums of algebras. Given algebras $(A, \mu)$ and $(B, \lambda)$, we may endow their module direct sum $A \oplus B$ with an algebra structure. Define

$$\nu : (A \oplus B) \otimes (A \oplus B) \longrightarrow A \oplus B$$

by

$$\nu((a_1, b_1) \otimes (a_2, b_2)) = (\mu(a_1 \otimes a_2), \lambda(b_1 \otimes b_2)).$$

$\nu$ is seen to be well defined after noting that the function

$$((a_1, b_1), (a_2, b_2)) \longmapsto (\mu(a_1 \otimes a_2), \lambda(b_1 \otimes b_2))$$

is bilinear. In this way, $(A \oplus B, \nu)$ is an algebra. This agrees with the usual meaning of the direct sum of two algebras using component-wise multiplication.

The above example illustrates how the direct sum of algebras can be constructed completely in terms of linear maps on tensor products. In the case that the index set $I$ is arbitrary, an arrow-theoretic argument (à la [9]) can be used to define component-wise multiplication

$$\nu : \left(\bigoplus_i A_i\right) \otimes \left(\bigoplus_i A_i\right) \longrightarrow \bigoplus_i A_i.$$

Theorem 3.1 *Let $(A_i, \mu_i)$ be $K$-algebras for $i \in I$ and let $(A, \nu)$ be their algebra direct sum. Then, $(A, \nu)$ is zero product determined if and only if $(A_i, \mu_i)$ is zero product determined for all $i \in I$.*

*Proof* We first note that $(\bigoplus_i A_i) \otimes (\bigoplus_i A_i) \cong \bigoplus_{i,j} A_i \otimes A_j$ by the correspondence

$$\sigma : (a_i)_i \otimes (b_i)_i \longmapsto (a_i \otimes b_j)_{i,j},$$

seen after making two applications of Proposition 2.1 in Chapter XVI of [10]. Then, $\nu$ factors through $\sigma$ as $\nu = \tilde{\nu} \circ \sigma$, where $\tilde{\nu} = \bigoplus_{i,j} \delta_{i,j} \mu_i$ ($\delta$ is the Kronecker delta). Because multiplication is defined component-wise, we have (after abusing the order of summands)

$$\sigma\,(\mathrm{Ker}\,\nu) = \mathrm{Ker}\,\tilde{\nu} = \bigoplus_i \mathrm{Ker}\,\mu_i \oplus \bigoplus_{i \neq j} A_i \otimes A_j. \qquad (\sharp)$$

Similarly, we have

$$\begin{aligned} T_\nu &= \{\,(a_i)_i \otimes (b_i)_i \,|\, \mu_i(a_i \otimes b_i) = 0 \text{ for all } i\} \\ \sigma\,(\langle T_\nu \rangle) &= \langle \sigma\,((a_i)_i \otimes (b_i)_i)\,|\mu_i(a_i \otimes b_i) = 0 \text{ for all } i \rangle \\ &= \left\langle \sum_{i,j} a_i \otimes b_j \,\middle|\, \mu_i(a_i \otimes b_i) = 0 \text{ for all } i \right\rangle \\ &= \bigoplus_i \langle T_{\mu_i} \rangle \oplus \bigoplus_{i \neq j} A_i \otimes A_j. \qquad (\flat) \end{aligned}$$

Comparing ($\sharp$) and ($\flat$), we see that $\mathrm{Ker}\,\nu = \langle T_\nu \rangle$ if and only if $\mathrm{Ker}\,\mu_i = \langle T_{\mu_i} \rangle$ for each $i$. Applying Theorem 2.3 completes the proof. □

## 4. Tensor products of algebras

As with direct sums, the $K$-module tensor product of two algebras may be endowed with a natural algebra structure. If the component algebras are zero product determined, then–under certain additional conditions–so is the tensor product. Theorem 2.3 elaborates.

We need the following two elementary results on tensor products of $K$-modules.

Proposition 4.1 *Let $K$ be a field. Let $f : A \longrightarrow C$ and $g : B \longrightarrow D$ be linear maps. Then*

$$\mathrm{Ker}\,(f \otimes g) = (\mathrm{Ker}\,f) \otimes B + A \otimes (\mathrm{Ker}\,g)\,.$$

*Proof* We have the $K$-vector space isomorphisms

$$\begin{aligned} &(A \otimes B)/\{(\mathrm{Ker}\,f) \otimes B + A \otimes (\mathrm{Ker}\,g)\} \\ &\quad \simeq (A/\,\mathrm{Ker}\,f) \otimes (B/\,\mathrm{Ker}\,g) \\ &\quad \simeq \mathrm{Im}(f) \otimes \mathrm{Im}(g) \;=\; \mathrm{Im}(f \otimes g) \\ &\quad \simeq (A \otimes B)/\,\mathrm{Ker}(f \otimes g). \end{aligned}$$

This proves Proposition 4.1. □

Proposition 4.2 *Let $A$, $B$, $C$, and $D$ be $K$-modules and let $f : A \longrightarrow C$ and $g : B \longrightarrow D$ be surjective $K$-linear maps. Then*

$$\operatorname{Ker}(f \otimes g) = (\operatorname{Ker} f) \otimes B + A \otimes (\operatorname{Ker} g).$$

*Proof* We have

$$\begin{aligned}(A \otimes B)/\{(\operatorname{Ker} f) \otimes B + A \otimes (\operatorname{Ker} g)\} \\ \simeq (A/\operatorname{Ker} f) \otimes (B/\operatorname{Ker} g) \\ \simeq C \otimes D \simeq (A \otimes B)/\operatorname{Ker}(f \otimes g).\end{aligned}$$

This proves Proposition 4.2. □

In Proposition 4.2, if $f$ or $g$ is not surjective, the equality of $\operatorname{Ker}(f \otimes g)$ may not hold. We provide the following example.

*Example* Let $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z} = \{\bar{0}, \bar{1}, \ldots, \overline{m-1}\}$. Consider the $\mathbb{Z}$-linear maps $f : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$, $f(\bar{1}) = \bar{1}$, and $g : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4$, $g(\bar{1}) = \bar{2}$. Then $f \otimes g$ sends $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ to $\mathbb{Z}_2 \otimes \mathbb{Z}_4$, and $(f \otimes g)(\bar{1} \otimes \bar{1}) = \bar{1} \otimes \bar{2} = \bar{2} \otimes \bar{1} = 0$ in $\mathbb{Z}_2 \otimes \mathbb{Z}_4$. We have $\operatorname{Ker}(f \otimes g) = \mathbb{Z}_2 \otimes \mathbb{Z}_2$. However, $\operatorname{Ker} f = \{\bar{0}\}$ and $\operatorname{Ker} g = \{\bar{0}\}$. So

$$\operatorname{Ker}(f \otimes g) \supsetneq (\operatorname{Ker} f) \otimes B + A \otimes (\operatorname{Ker} g).$$

We now switch our attention from tensor product of general $K$-modules to tensor product of $K$-algebras. Given $K$-algebras $(A, \mu)$ and $(B, \lambda)$, their $K$-module tensor product $A \otimes B$ may be endowed with a component-wise multiplication map $\kappa : A \otimes B \otimes A \otimes B \longrightarrow A \otimes B$ given by

$$\kappa(a_1 \otimes b_1 \otimes a_2 \otimes b_2) = \mu(a_1 \otimes a_2) \otimes \lambda(b_1 \otimes b_2).$$

$(A \otimes B, \kappa)$ is then a $K$-algebra and is called the algebra tensor product of the $K$-algebras $A$ and $B$.

Theorem 4.3 *Let $(A, \mu)$ and $(B, \lambda)$ be zero product determined $K$-algebras satisfying the added condition that*

$$\operatorname{Ker}(\mu \otimes \lambda) = (\operatorname{Ker} \mu) \otimes (B \otimes B) + (A \otimes A) \otimes (\operatorname{Ker} \lambda).$$

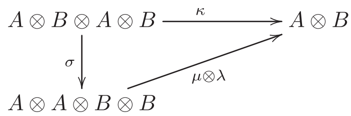
*Then, their algebra tensor product $(A \otimes B, \kappa)$ is zero product determined. In particular, $(A \otimes B, \kappa)$ is zero product determined when:*

(1) *$K$ is a field, or*
(2) $\operatorname{Im} \mu = A$ *and* $\operatorname{Im} \lambda = B$.

*Proof* Let $\sigma$ be the obvious isomorphism

$$\sigma : A \otimes B \otimes A \otimes B \longrightarrow A \otimes A \otimes B \otimes B.$$

Then, $\kappa$ factors through $\sigma$ as $\kappa = (\mu \otimes \lambda) \circ \sigma$.




The equality

$$\text{Ker}\,(\mu \otimes \lambda) = (\text{Ker}\,\mu) \otimes (B \otimes B) + (A \otimes A) \otimes (\text{Ker}\,\lambda)$$

immediately shows that $\text{Ker}\,(\mu \otimes \lambda) = \langle T_{\mu\otimes\lambda} \rangle$, whose inverse image under $\sigma$ is

$$\text{Ker}\,\kappa = \langle T_\kappa \rangle.$$

By Theorem 2.3, $(A \otimes B, \kappa)$ is zero product determined. Applying Proposition 4.1 or 4.2 where appropriate completes the proof. □

*Example* It was shown that $\mathbb{C}[x, y]$ (the free commutative $\mathbb{C}$-algebra on two generators) is not zero product determined. In fact, Theorem 4.3 implies that $\mathbb{C}[x]$ is not zero product determined. Assume for the contrary that $\mathbb{C}[x]$ is zero product determined. $\mathbb{C}[x, y] = \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y]$, and the theorem would give that $\mathbb{C}[x, y]$ is zero product determined–a contradition.

Theorem 4.3 may be extended to the tensor product of finitely many algebras as follow:

THEOREM 4.4 *Let $(A_i, \mu_i)$, $i = 1, 2, \ldots, n$, be zero product determined $K$-algebras. Let $\left(\bigotimes_{i=1}^n A_i, \kappa\right)$ be the algebra tensor product of $(A_i, \mu_i)$, $i = 1, 2, \ldots, n$. If one of the following conditions holds:*

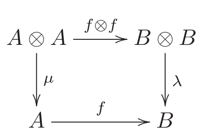
(1) *$K$ is a field, or*
(2) $\text{Im}\,\mu_i = A_i$ *for* $i = 1, 2, \ldots n$,

*then $\left(\bigotimes_{i=1}^n A_i, \kappa\right)$ is zero product determined.*

*Proof* For either case, we use Theorem 4.3 and induction to complete the proof. □

## 5. Homomorphic images of algebras

We next examine the relationship between homomorphic images of algebras and the property of being zero product determined. For algebras $(A, \mu)$ and $(B, \lambda)$, an algebra homomorphism $f : A \longrightarrow B$ is simply a linear map such that the diagram commutes.




We begin with a general lemma.

LEMMA 5.1 *Let $f : A \to B$ be a $K$-algebra homomorphism from $(A, \mu)$ to $(B, \lambda)$. Then,*

$$\operatorname{Ker} f \cap \operatorname{Im} \mu = \mu\left((f \otimes f)^{-1}(\operatorname{Ker} \lambda)\right).$$

*Proof* By $f \circ \mu = \lambda \circ (f \otimes f)$, we get $\operatorname{Ker} \mu \subseteq (f \otimes f)^{-1}(\operatorname{Ker} \lambda)$. Then

$$\begin{aligned}
\mu\left(\sum_i a_{i1} \otimes a_{i2}\right) \in \operatorname{Ker} f \cap \operatorname{Im} \mu &\iff f \circ \mu\left(\sum_i a_{i1} \otimes a_{i2}\right) = 0 \\
&\iff \lambda \circ (f \otimes f)\left(\sum_i a_{i1} \otimes a_{i2}\right) = 0 \\
&\iff \sum_i a_{i1} \otimes a_{i2} \in (f \otimes f)^{-1}(\operatorname{Ker} \lambda) \\
&\iff \mu\left(\sum_i a_{i1} \otimes a_{i2}\right) \in \mu\left((f \otimes f)^{-1}(\operatorname{Ker} \lambda)\right).
\end{aligned}$$

□

THEOREM 5.2 *Let $(A, \mu)$ be a zero product determined $K$-algebra, and $f : A \to B$ a surjective $K$-algebra homomorphism from $(A, \mu)$ to $(B, \lambda)$. Then, $(B, \lambda)$ is zero product determined if and only if*

$$\operatorname{Ker} f \cap \operatorname{Im} \mu \subseteq \mu\left((f \otimes f)^{-1}(\langle T_\lambda \rangle)\right).$$

*Remark* By Lemma 5.1, the inclusion in the last formula may be replaced by equality and the statement still holds.

*Proof* We proceed in two steps:

(1) If $(B, \lambda)$ is zero product determined, then $\operatorname{Ker} \lambda = \langle T_\lambda \rangle$. Lemma 5.1 immediately implies that $\operatorname{Ker} f \cap \operatorname{Im} \mu = \mu\left((f \otimes f)^{-1}(\langle T_\lambda \rangle)\right)$.

(2) Suppose $\operatorname{Ker} f \cap \operatorname{Im} \mu \subseteq \mu\left((f \otimes f)^{-1}(\langle T_\lambda \rangle)\right)$. To prove that $(B, \lambda)$ is zero product determined, it suffices to show that $\operatorname{Ker} \lambda \subseteq \langle T_\lambda \rangle$. By Lemma 5.1,

$$\mu\left((f \otimes f)^{-1}(\operatorname{Ker} \lambda)\right) \subseteq \mu\left((f \otimes f)^{-1}(\langle T_\lambda \rangle)\right).$$

So,

$$(f \otimes f)^{-1}(\operatorname{Ker} \lambda) \subseteq (f \otimes f)^{-1}(\langle T_\lambda \rangle) + \operatorname{Ker} \mu. \tag{$\natural$}$$

Since $(A, \mu)$ is zero product determined, $\operatorname{Ker} \mu = \langle T_\mu \rangle$. Moreover, for any $a' \otimes a'' \in T_\mu$,

$$0 = f(\mu(a' \otimes a'')) = \lambda(f(a') \otimes f(a'')).$$

Therefore, $f(a') \otimes f(a'') \in T_\lambda$. It shows that $T_\mu \subseteq (f \otimes f)^{-1}(T_\lambda)$ and so $\langle T_\mu \rangle \subseteq (f \otimes f)^{-1}(\langle T_\lambda \rangle)$. Then $\natural$ becomes

$$(f \otimes f)^{-1}(\operatorname{Ker} \lambda) \subseteq (f \otimes f)^{-1}(\langle T_\lambda \rangle).$$

The surjectivity of $f$ implies the surjectivity of $f \otimes f$. Therefore, $\operatorname{Ker}\lambda \subseteq \langle T_\lambda \rangle$ as desired. □

Theorem 5.2 implies the following two corollaries.

COROLLARY 5.3 *Let $f : A \to B$ be a surjective $K$-algebra homomorphism from $(A, \mu)$ to $(B, \lambda)$. If $(A, \mu)$ is zero product determined and satisfies*

$$\mu\left(A \otimes (\operatorname{Ker} f) + (\operatorname{Ker} f) \otimes A\right) = \operatorname{Ker} f \cap \operatorname{Im} \mu$$

*(for example, if A has unity), then $(B, \lambda)$ is zero product determined.*

*Proof* Obviously,

$$A \otimes (\operatorname{Ker} f) + (\operatorname{Ker} f) \otimes A \subseteq (f \otimes f)^{-1} (\langle T_\lambda \rangle),$$

and we apply Theorem 5.2. □

COROLLARY 5.4 *Let $(A, \mu)$ be a $K$-algebra with the property that for each $a \in A$, there are $a', a'' \in A$ where $\mu(a' \otimes a'') = a$ (for example, if A has unity). Then, if A is zero product determined, so are all of its homomorphic images.*

*Proof* Suppose the hypotheses, and suppose that we are given a $K$-algebra $(B, \lambda)$ and a surjective $K$-algebra homomorphism $f$ from $A$ onto $B$. Given $a \in \operatorname{Ker} f$, we can write $a = \mu(a' \otimes a'')$ for some $a', a'' \in A$. Then,

$$0 = f(a) = f(\mu(a' \otimes a'')) = \lambda(f(a') \otimes f(a'')).$$

This is to say that $f(a') \otimes f(a'') \in \langle T_\lambda \rangle$. Therefore,

$$\operatorname{Ker} f \subseteq \mu\left((f \otimes f)^{-1} (\langle T_\lambda \rangle)\right),$$

and we apply Theorem 5.2. □

As a simple example, if $I$ is an ideal of $K$, then $K/I$ as a $K$-algebra is zero product determined.

*Example* Suppose $A$ is zero product determined and $B$ is not zero product determined. Then, $A \oplus B$ is not zero product determined by Theorem 3.1. We make note of the following observations concerning algebra homomorphisms:

(1) Claim: *A surjective algebra homomorphism may send a non-zero product determined algebra to either a zero product determined algebra or a non-zero product determined algebra.* For examples, consider the projection maps $P_A : A \oplus B \longrightarrow A$ and $P_B : A \oplus B \longrightarrow B$.
(2) Claim: *A non-surjective algebra homomorphism may send a zero product determined algebra to a non-zero product determined algebra.* For example, consider the inclusion map from $A$ to $A \oplus B$.

## 6. Ladder matrix algebras

Let $A$ be a $K$-algebra with unity $1_A$, and $M_n(A)$ the $n \times n$ matrices of entries in $A$. Brešar, Grašič, and Sánchez Ortega proved the following result in [1]: when $n \geq 2$, $M_n(A)$ with matrix multiplication is zero product determined. We extend this result by investigate the zero product determined property relative to algebras comprised of matrices that we will call ladder shape matrices. Let $E_{ij} \in M_n(A)$ denote the matrix with the only non-zero entry $1_A$ in the $(i, j)$ position.

*Definition 6.1* Given an ordered set of integer pairs

$$\mathcal{L} := \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\},$$

where $k \in \mathbb{Z}^+$, $1 \leq i_1 < \cdots < i_k \leq n$, $1 \leq j_1 < \cdots < j_k \leq n$, we call $\mathcal{L}$ a $k$ **step ladder of size** $n$ or simply a **ladder**. Define the **index set of ladder** $\mathcal{L}$ by

$$\Sigma_{\mathcal{L}} := \{(i, j) \in \{1, 2, \ldots, n\}^2 \mid i \leq i_t \text{ and } j_t \leq j \text{ for some } (i_t, j_t) \in \mathcal{L}\}.$$

Define the **set of $\mathcal{L}$ ladder matrices** by

$$M_{\mathcal{L}}(A) := \sum_{(i,j)\in\Sigma_{\mathcal{L}}} AE_{ij}.$$

In brief, $M_{\mathcal{L}}(A)$ contains the matrices in $M_n(A)$ with every non-zero entry locating on the upper right direction of certain $(i_t, j_t)$.

In the following discussion, we assume that $\mathcal{L}$ and $\mathcal{L}'$ are two ladders of size $n$ as follow:

$$\mathcal{L} = \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\},$$
$$\mathcal{L}' = \{(i'_1, j'_1), (i'_2, j'_2), \ldots, (i'_{k'}, j'_{k'})\}.$$

THEOREM 6.2 *The product of ladder matrices satisfies that*

$$M_{\mathcal{L}}(A)M_{\mathcal{L}'}(A) = M_{\mathcal{L}''}(A),$$

*where*

$$\mathcal{L}'' = \left\{ (i_t, j'_s) \mid t = \max\{u \mid j_u \leq i'_s\},\ s = \min\{v \mid j_t \leq i'_v\} \right\}.$$

*In particular, $\mathcal{L}''$ is a ladder with step $k'' \leq \min\{k, k'\}$, and every pair $(i_t, j'_s) \in \mathcal{L}''$ satisfies $j_t \leq i'_s$.*

To prove the above theorem, we need the following results about $\mathcal{L}''$.

LEMMA 6.3 *Let $\mathcal{L}$ and $\mathcal{L}'$ be given above.*

(1) *(Algorithm) for any $s' \in \{1, 2, \ldots, k'\}$, let*

$$t := \max\{u \mid j_u \leq i'_{s'}\}, \quad s := \min\{v \mid j_t \leq i'_v\}.$$

*Then, $(i_t, j'_s)$ is an element of the set $\mathcal{L}''$ defined in Theorem 6.2. In other words, $t = \max\{u \mid j_u \leq i'_s\}$.*

(2) *Any element of the set $\mathcal{L}''$ may be obtained by Algorithm* (1).

(3) *If* $(i_{t_1}, j'_{s_1})$ *and* $(i_{t_2}, j'_{s_2})$ *are distinct elements of the set* $\mathcal{L}''$, *and* $t_1 \le t_2$, *then* $t_1 < t_2$ *and* $s_1 < s_2$.

(4) *For* $(t, s) \in \{1, 2, \dots, k\} \times \{1, 2, \dots, k'\}$, $(i_t, j'_s) \in \Sigma_{\mathcal{L}''}$ *if and only if* $j_t \le i'_s$.

*Proof* We prove the claims in order:

(1) The definition of $t$ says that

$$j_t \le i'_{s'} < j_{t+1} \text{ (or } n+1 \text{ if } t = k).$$

The definition of $s$ says that

$$i'_{s-1} \text{ (or 0 if } s = 1) < j_t \le i'_s.$$

Hence, $s \le s'$ and thus,

$$j_t \le i'_s \le i'_{s'} < j_{t+1} \text{ (or } n+1 \text{ if } t = k).$$

It proves that $t = \max\{u \mid j_u \le i'_s\}$.

(2) If $(i_t, j'_s) \in \mathcal{L}''$, then by definition

$$t = \max\{u \mid j_u \le i'_s\}, \quad s = \min\{v \mid j_t \le i'_v\}.$$

Starting at $s$ and applying Algorithm (1), we get the pair $(i_t, j'_s)$.

(3) Suppose $(i_{t_1}, j'_{s_1})$ and $(i_{t_2}, j'_{s_2})$ are distinct elements of $\mathcal{L}''$, and $t_1 \le t_2$. If $t_1 = t_2$, then $s_1 = s_2$ by the definition of $\mathcal{L}''$. A contradiction! So $t_1 < t_2$. Then, $j_{t_1} < j_{t_2}$. Since

$$s_1 = \min\{v \mid j_{t_1} \le i'_v\} \quad \text{and} \quad s_2 = \min\{v \mid j_{t_2} \le i'_v\},$$

we have $s_1 \le s_2$. Then $s_1 < s_2$ by a similar reasoning.

(4) Suppose $(t, s) \in \{1, 2, \dots, k\} \times \{1, 2, \dots, k'\}$ satisfies that $j_t \le i'_s$. We construct $(i_{t_1}, j'_{s_1}) \in \mathcal{L}''$ by Algorithm (1) as follow:

$$t_1 := \max\{u \mid j_u \le i'_s\}, \quad s_1 := \min\{v \mid j_{t_1} \le i'_v\}.$$

Then $j_t \le j_{t_1} \le i'_s$, so that $j_{t_1} \le i'_{s_1} \le i'_s$. Therefore, $i_t \le i_{t_1}$ and $j'_{s_1} \le j'_s$. This shows that $(i_t, j'_s) \in \Sigma_{\mathcal{L}''}$.

Conversely, suppose $(i_t, j'_s) \in \Sigma_{\mathcal{L}''}$. There exists $(i_{t_1}, j'_{s_1}) \in \mathcal{L}''$, such that

$$i_t \le i_{t_1}, \qquad j'_{s_1} \le j'_s.$$

Hence, $t \le t_1$ and $s_1 \le s$. Algorithm (1) shows that $j_{t_1} \le i'_{s_1}$. Therefore, $j_t \le j_{t_1} \le i'_{s_1} \le i'_s$. □

*Proof of Theorem 6.2* Let $(i, j') \in \Sigma_{\mathcal{L}''}$. Then $i \le i_t$ and $j'_s \le j'$ for certain $(i_t, j'_s) \in \mathcal{L}''$. By the definition of $\mathcal{L}''$, we have $j_t \le i'_s$. By the definition of $\Sigma_{\mathcal{L}}$ and $\Sigma_{\mathcal{L}'}$, we get $(i, j_t) \in \Sigma_{\mathcal{L}}$ and $(j_t, j') \in \Sigma_{\mathcal{L}'}$. Therefore, $E_{ij_t} \in M_{\mathcal{L}}(A)$, $E_{j_t, j'} \in M_{\mathcal{L}'}(A)$ and $E_{ij'} = E_{ij_t} E_{j_t j'} \in M_{\mathcal{L}}(A) M_{\mathcal{L}'}(A)$. This shows that $M_{\mathcal{L}''}(A) \subseteq M_{\mathcal{L}}(A) M_{\mathcal{L}'}(A)$.

Now suppose $X = \left(x_{ij}\right)_{n \times n}$ is a matrix in $M_{\mathcal{L}}(A) M_{\mathcal{L}'}(A)$, that is,

$$X = \sum_{r=1}^{m} Y_r Z_r, \quad Y_r \in M_{\mathcal{L}}(A),\ Z_r \in M_{\mathcal{L}'}(A),\ r = 1, 2, \dots, m.$$

If $x_{pq} \neq 0$ for certain $(p, q)$, then the $(p, q)$ entry of certain $Y_r Z_r$ is non-zero. Set $Y_r = (y_{ij})_{n\times n}$, $Z_r = (z_{ij})_{n\times n}$. The $(p, q)$ entry of $Y_r Z_r$ is $\sum_{i=1}^{n} y_{pi} z_{iq}$. There exists $i$ such that $y_{pi} \neq 0$ and $z_{iq} \neq 0$. Then $(p, i) \in \Sigma_{\mathcal{L}}$ and $(i, q) \in \Sigma_{\mathcal{L}'}$. There are $(i_t, j_t) \in \mathcal{L}$ and $(i'_s, j'_s) \in \mathcal{L}'$, such that

$$p \leq i_t, \quad j_t \leq i, \qquad i \leq i'_s, \quad j'_s \leq q.$$

So $j_t \leq i'_s$. Thus, $(i_t, j'_s) \in \Sigma_{\mathcal{L}''}$ according to Lemma 6.3(4). Then $(p, q) \in \Sigma_{\mathcal{L}''}$ by $p \leq i_t$ and $j'_s \leq q$. Therefore, $M_{\mathcal{L}}(A)M_{\mathcal{L}'}(A) \subseteq M_{\mathcal{L}''}(A)$.

Overall, we have proved that $M_{\mathcal{L}}(A)M_{\mathcal{L}'}(A) = M_{\mathcal{L}''}(A)$. □

*Definition 6.4* A ladder $\mathcal{L} = \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ is called a **block upper triangular ladder**, if $i_t < j_{t+1}$ for $t = 1, 2, \ldots, k-1$ (Figure 1).

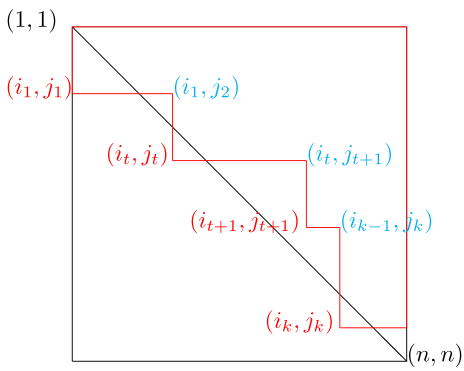



Figure 1. A block upper triangular ladder matrix.

For example, the set of block upper triangular matrices of $M_n(A)$ with respect to a partition $n_1, n_2, \ldots, n_k$ of $n$ coincides with $M_{\mathcal{L}}(A)$ for

$$\mathcal{L} = \{(n_1, 1), (n_1 + n_2, n_1 + 1), (n_1 + n_2 + n_3, n_1 + n_2 + 1), \ldots\}.$$

Such an $\mathcal{L}$ is a block upper triangular ladder.

THEOREM 6.5 *$M_{\mathcal{L}}(A)$ with matrix product forms a $K$-algebra if and only if $\mathcal{L}$ is a block upper triangular ladder.*

*Proof* Let $\mathcal{L} = \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ and $M_{\mathcal{L}}(A)M_{\mathcal{L}}(A) = M_{\mathcal{L}''}(A)$. Then $M_{\mathcal{L}}(A)$ is a $K$-algebra if and only if $M_{\mathcal{L}''}(A) \subseteq M_{\mathcal{L}}(A)$.

Suppose $\mathcal{L}$ is block upper triangular so that $i_t < j_{t+1}$ for $t = 1, 2, \ldots, k-1$. We prove that $\mathcal{L}'' \subseteq \Sigma_{\mathcal{L}}$, so that $M_{\mathcal{L}''}(A) \subseteq M_{\mathcal{L}}(A)$. Given $(i_t, j_s) \in \mathcal{L}''$, we have $j_t \leq i_s$. Since $\mathcal{L}$ is block upper triangular, $i_s < j_{s+1}$. Hence, $t \leq s$ and $j_t \leq j_s$. By $(i_t, j_t) \in \mathcal{L}$, we have $(i_t, j_s) \in \Sigma_{\mathcal{L}}$. Therefore, $\mathcal{L}'' \subseteq \Sigma_{\mathcal{L}}$.

Now suppose $\mathcal{L}$ is not block upper triangular so that $j_{t+1} \leq i_t$ for certain $t \in \{1, 2, \ldots, k-1\}$. Then $(i_{t+1}, j_t) \in \Sigma_{\mathcal{L}''}$ by Lemma 6.3(4). However, $j_t < j_{t+1}$, and we see that $(i_{t+1}, j_t) \notin \Sigma_{\mathcal{L}}$. Therefore, $M_{\mathcal{L}''}(A) \nsubseteq M_{\mathcal{L}}(A)$. □

THEOREM 6.6 *Let $(A, \mu)$ be a $K$-algebra with unity $1_A$. Let $\mathcal{L} = \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ be a block upper triangular ladder of size $n$. Then, $M_{\mathcal{L}}(A)$ with induced matrix product $\hat{\mu}$ is a zero product determined $K$-algebra, provided that one of the following conditions holds:*

(1) *$(A, \mu)$ is zero product determined.*
(2) *$\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_k\}$ have no intersection.*

*Proof* For convenience, we use the product operation on $A$ to denote $\mu$ , and the matrix product operation on $M_{\mathcal{L}}(A)$ to denote $\hat{\mu}$.

Let $M_{\mathcal{L}}(A)M_{\mathcal{L}}(A) = M_{\mathcal{L}''}(A)$. To prove that $(M_{\mathcal{L}}(A), \hat{\mu})$ is zero product determined, it suffices to prove that $\operatorname{Ker} \hat{\mu} \subseteq \langle T_{\hat{\mu}} \rangle$.

Given $\sum_{r=1}^{m} M^{(r)} \otimes N^{(r)} \in \operatorname{Ker} \hat{\mu}$, suppose

$$M^{(r)} = \sum_{(i,j)\in\Sigma_{\mathcal{L}}} m_{ij}^{(r)} E_{ij}, \qquad N^{(r)} = \sum_{(i,j)\in\Sigma_{\mathcal{L}}} n_{ij}^{(r)} E_{ij}.$$

Then,

$$\hat{\mu}\left(\sum_{r=1}^{m} M^{(r)} \otimes N^{(r)}\right) = \sum_{r=1}^{m} M^{(r)} N^{(r)} = 0.$$

For any $(p, q) \in \Sigma_{\mathcal{L}''}$,

$$\sum_{r=1}^{m} \sum_{\substack{\ell \\ (p,\ell),(\ell,q)\in\Sigma_{\mathcal{L}}}} m_{p\ell}^{(r)} n_{\ell q}^{(r)} = 0. \tag{6.1}$$

So

$$\sum_{r=1}^{m} \sum_{\substack{\ell \\ (p,\ell),(\ell,q)\in\Sigma_{\mathcal{L}}}} m_{p\ell}^{(r)} \otimes n_{\ell q}^{(r)} \in \operatorname{Ker} \mu. \tag{6.2}$$

Explicit computation shows that

$$\begin{aligned}
\sum_{r=1}^{m} M^{(r)} \otimes N^{(r)} &= \sum_{r=1}^{m} \left(\sum_{(i,j)\in\Sigma_{\mathcal{L}}} m_{ij}^{(r)} E_{ij}\right) \otimes \left(\sum_{(i,j)\in\Sigma_{\mathcal{L}}} n_{ij}^{(r)} E_{ij}\right) \\
&= \sum_{r=1}^{m} \left( \sum_{(p,\ell),(\ell,q)\in\Sigma_{\mathcal{L}}} m_{p\ell}^{(r)} E_{p\ell} \otimes n_{\ell q}^{(r)} E_{\ell q} + \sum_{\substack{(p,s),(t,q)\in\Sigma_{\mathcal{L}} \\ s\neq t}} m_{ps}^{(r)} E_{ps} \otimes n_{tq}^{(r)} E_{tq} \right) \\
&\in \sum_{(p,q)\in\Sigma_{\mathcal{L}''}} \left( \sum_{r=1}^{m} \sum_{\substack{\ell \\ (p,\ell),(\ell,q)\in\Sigma_{\mathcal{L}}}} m_{p\ell}^{(r)} E_{p\ell} \otimes n_{\ell q}^{(r)} E_{\ell q} \right) + \langle T_{\hat{\mu}} \rangle
\end{aligned}$$

For any $(p,q) \in \Sigma_{\mathcal{L}''}$, we can choose a representative $\ell_{pq}$ such that $(p,\ell_{pq}), (\ell_{pq},q) \in \Sigma_{\mathcal{L}}$. Then,

$$\begin{aligned}
&\sum_{r=1}^{m} \sum_{\substack{\ell \\ (p,\ell),(\ell,q)\in\Sigma_{\mathcal{L}}}} m_{p\ell}^{(r)} E_{p\ell} \otimes n_{\ell q}^{(r)} E_{\ell q} \\
&\quad = \sum_{r=1}^{m} \sum_{\substack{\ell \\ (p,\ell),(\ell,q)\in\Sigma_{\mathcal{L}}}} \left(m_{p\ell}^{(r)} \otimes n_{\ell q}^{(r)}\right) \left(E_{p\ell} \otimes E_{\ell q} - E_{p\ell_{pq}} \otimes E_{\ell_{pq}q}\right) \\
&\qquad + \left( \sum_{r=1}^{m} \sum_{\substack{\ell \\ (p,\ell),(\ell,q)\in\Sigma_{\mathcal{L}}}} m_{p\ell}^{(r)} \otimes n_{\ell q}^{(r)} \right) \left(E_{p\ell_{pq}} \otimes E_{\ell_{pq}q}\right).
\end{aligned} \tag{6.3}$$

Each summand of the first term of (6.3) satisfies that

$$\begin{aligned}
&\left(m_{p\ell}^{(r)} \otimes n_{\ell q}^{(r)}\right) \left(E_{p\ell} \otimes E_{\ell q} - E_{p\ell_{pq}} \otimes E_{\ell_{pq}q}\right) \\
&\quad = \left(m_{p\ell}^{(r)} \otimes n_{\ell q}^{(r)}\right) \left[\left(E_{p\ell} - E_{p\ell_{pq}}\right) \otimes \left(E_{\ell q} + E_{\ell_{pq}q}\right) - E_{p\ell} \otimes E_{\ell_{pq}q} + E_{p\ell_{pq}} \otimes E_{\ell q}\right] \\
&\quad \in \langle T_{\hat{\mu}} \rangle.
\end{aligned}$$

Next, we show that the second term of (6.3) also belongs to $\langle T_{\hat{\mu}} \rangle$ when one of the conditions in Theorem 6.6 is satisfied.

(1) $(A,\mu)$ is zero product determined. Then, $\operatorname{Ker}\mu = \langle T_\mu \rangle$. According to (6.2),

$$\begin{aligned}
&\left( \sum_{r=1}^{m} \sum_{\substack{\ell \\ (p,\ell),(\ell,q)\in\Sigma_{\mathcal{L}}}} m_{p\ell}^{(r)} \otimes n_{\ell q}^{(r)} \right) \left(E_{p\ell_{pq}} \otimes E_{\ell_{pq}q}\right) \\
&\in \langle T_\mu \rangle \left(E_{p\ell_{pq}} \otimes E_{\ell_{pq}q}\right) \subseteq \langle T_{\hat{\mu}} \rangle.
\end{aligned}$$

(2) $\{i_1, i_2, \dots, i_k\}$ and $\{j_1, j_2, \dots, j_k\}$ have no intersection. We use an approach similar to the proof of [1, Theorem 2.1]. Define

$$t = \min\{v \mid p \le i_v\}, \qquad s = \max\{u \mid j_u \le q\}.$$

Let $\gamma$ be any integer such that $(p,\gamma), (\gamma,q) \in \Sigma_{\mathcal{L}}$. Then

$$\begin{aligned}
i_{t-1} \text{ (or 0 if } t = 1) < p \le i_t \quad &\Longrightarrow \quad j_t \le \gamma, \\
j_s \le q < j_{s+1} \text{ (or } n+1 \text{ if } s = k) \quad &\Longrightarrow \quad \gamma \le i_s.
\end{aligned}$$

So, $j_t \le \gamma \le i_s$. Conversely, if integer $\gamma$ satisfies that $j_t \le \gamma \le i_s$, then $(p,\gamma), (\gamma,q) \in \Sigma_{\mathcal{L}}$.

By assumption $j_t \neq i_s$. So, $j_t < i_s$. Let $\gamma = \ell_{pq}$. Let $\delta$ be an integer such that $j_t \leq \delta \leq i_s$ and $\gamma \neq \delta$. Then for any elements $a, b \in A$,

$$\begin{aligned}(a \otimes b)\left(E_{p\gamma} \otimes E_{\gamma q}\right) - (ab \otimes 1_A)\left(E_{p\delta} \otimes E_{\delta q}\right) \\ = \left(aE_{p\gamma} + abE_{p\delta}\right) \otimes \left(bE_{\gamma q} - E_{\delta q}\right) \\ -abE_{p\delta} \otimes bE_{\gamma q} + aE_{p\gamma} \otimes E_{\delta q} \\ \in \langle T_{\hat{\mu}} \rangle.\end{aligned}$$

Hence,

$$\begin{aligned}&\left(\sum_{r=1}^{m} \sum_{\substack{\ell \\ (p,\ell),(\ell,q)\in\Sigma_{\mathcal{L}}}} m_{p\ell}^{(r)} \otimes n_{\ell q}^{(r)}\right)\left(E_{p\ell_{pq}} \otimes E_{\ell_{pq}q}\right) \\ &\in \left(\sum_{r=1}^{m} \sum_{\substack{\ell \\ (p,\ell),(\ell,q)\in\Sigma_{\mathcal{L}}}} m_{p\ell}^{(r)} n_{\ell q}^{(r)} \otimes 1_A\right)\left(E_{p\delta} \otimes E_{\delta q}\right) + \langle T_{\hat{\mu}} \rangle = \langle T_{\hat{\mu}} \rangle\end{aligned}$$

according to (6.1).

In both cases, we succeed in proving that $\sum_{r=1}^{m} M^{(r)} \otimes N^{(r)} \in \langle T_{\hat{\mu}} \rangle$. Therefore, $\operatorname{Ker} \hat{\mu} \subseteq \langle T_{\hat{\mu}} \rangle$ as desired. □

For examples, the matrix algebras of block upper triangular matrices, and the matrix algebra of strictly upper triangular matrices, over a zero product determined $K$-algebra $A$ with unity, are zero product determined.

Theorem 6.6 extends the result of [1, Theorem 2.1]. Moreover, Theorem 6.6 is sharp, according to the following theorem.

THEOREM 6.7 *Let $(A, \mu)$ be any non-zero-product-determined $K$-algebra with unity $1_A$. Let $\mathcal{L} = \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ be any block upper triangular ladder of size $n$. If $i_s = j_t$ for some indices $s$ and $t$, then $M_{\mathcal{L}}(A)$ with induced matrix product $\hat{\mu}$ is not a zero product determined $K$-algebra.*

*Proof* We use the product on $A$ to denote $\mu$, and the matrix product on $M_{\mathcal{L}}(A)$ to denote $\hat{\mu}$. Since $(A, \mu)$ is not zero product determined, there exists

$$\sum_{r=1}^{m} a^{(r)} \otimes b^{(r)} \in \operatorname{Ker} \mu - \langle T_{\mu} \rangle.$$

Suppose $M_{\mathcal{L}}(A)M_{\mathcal{L}}(A) = M_{\mathcal{L}''}(A)$. Then, $(i_t, j_s) \in \mathcal{L}''$ by $j_t = i_s$ and Lemma 6.3(1). Denote the matrices

$$M^{(r)} = a^{(r)} E_{i_t, j_t}, \quad N^{(r)} = b^{(r)} E_{i_s, j_s} = b^{(r)} E_{j_t, j_s}, \quad r = 1, 2, \ldots, m.$$

Then, $M^{(r)}, N^{(r)} \in M_{\mathcal{L}}(A)$ and $\sum_{r=1}^{m} M^{(r)}N^{(r)} = 0$, which shows that

$$\sum_{r=1}^{m} M^{(r)} \otimes N^{(r)} = \left(\sum_{r=1}^{m} a^{(r)} \otimes b^{(r)}\right)\left(E_{i_t,j_t} \otimes E_{j_t,j_s}\right) \in \operatorname{Ker}\hat{\mu}.$$

Now we claim that $\sum_{r=1}^{m} M^{(r)} \otimes N^{(r)} \notin \langle T_{\hat{\mu}}\rangle$. Suppose on the contrary,

$$\sum_{r=1}^{m} M^{(r)} \otimes N^{(r)} = \sum_{w=1}^{m'} M'^{(w)} \otimes N'^{(w)}$$

for some $M'^{(w)} \otimes N'^{(w)} \in T_{\hat{\mu}}$, $w = 1, 2, \ldots, m'$, $m' \in \mathbb{Z}^+$. Then, $M'^{(w)}N'^{(w)} = 0$. The only index $\gamma$ such that $(i_t, \gamma), (\gamma, j_s) \in \Sigma_{\mathcal{L}}$ is $\gamma = j_t = i_s$. Let $m^{(w)}$ denote the $(i_t, j_t)$ entry of $M'^{(w)}$, and $n^{(w)}$ the $(j_t, j_s)$ entry of $N'^{(w)}$. The $(i_t, j_s)$ entry of $M'^{(w)}N'^{(w)}$ is exactly $m^{(w)}n^{(w)} = 0$, which shows that $m^{(w)} \otimes n^{(w)} \in T_{\mu}$.

View $M_{\mathcal{L}}(A)$ as a free $A$-bimodule with the basis $\{E_{ij} \mid (i, j) \in \Sigma_{\mathcal{L}}\}$. Then $M_{\mathcal{L}}(A) \otimes_A M_{\mathcal{L}}(A)$ is a free $A$-bimodule with the basis $\{E_{ij} \otimes_A E_{pq} \mid (i, j), (p, q) \in \Sigma_{\mathcal{L}}\}$. The $K$-module monomorphism $K \to A$ by $k \mapsto k1_A$ induces a $K$-module epimorphism

$$M_{\mathcal{L}}(A) \otimes M_{\mathcal{L}}(A) \to M_{\mathcal{L}}(A) \otimes_A M_{\mathcal{L}}(A).$$

Therefore, the elements of $\{E_{ij} \otimes E_{pq} \mid (i, j), (p, q) \in \Sigma_{\mathcal{L}}\}$ in $M_{\mathcal{L}}(A) \otimes M_{\mathcal{L}}(A)$ are $K$-linearly independent. The coefficient of $E_{i_t,j_t} \otimes E_{j_t,j_s}$ in the expression of $\sum_{w=1}^{m'} M'^{(w)} \otimes N'^{(w)}$ is

$$\sum_{w=1}^{m'} m^{(w)} \otimes n^{(w)} = \sum_{r=1}^{m} a^{(r)} \otimes b^{(r)}.$$

It shows that $\sum_{r=1}^{m} a^{(r)} \otimes b^{(r)} \in \langle T_{\mu}\rangle$, a contradiction to our assumption.

Overall, we have $\operatorname{Ker}\hat{\mu} \neq \langle T_{\hat{\mu}}\rangle$, and thus $(M_{\mathcal{L}}(A), \hat{\mu})$ is not zero product determined. □

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