Upper Triangular Ladder Matrix Algebras

A Preliminary Report

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Upper Triangular Ladder Matrix Algebras

- A *k-step ladder on n* is a set

$$\mathcal{L} = \{(i_1, j_1), ..., (i_k, j_k)\}$$

with $1 \leq i_1 < i_2 < ... < i_k \leq n$ and $1 \leq j_1 < j_2 < ... < j_k \leq n$.

- The *ladder matrices on $\mathcal{L}$* is the space

$$M_{\mathcal{L}} := \operatorname{Span}\{e_{i,j}\,;\,(i,j) \in I\} \subseteq M_F^{n\times n}$$

where

$$I = \{(i,j)\,;\,\exists(i_t, j_t) \in \mathcal{L}, i \leq i_t, j \geq j_t\}$$

- A ladder $\mathcal{L}$ is called *upper triangular* when $i_t < j_{t+1}$ for $t = 1, 2, ..., k-1$.

Example

Let $\mathcal{L} = \{(3,2), (6,5)\}$,
a 2-step ladder on 6.

$$M_{\mathcal{L}} = \left\{ \begin{pmatrix} 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix} \right\}$$

Upper Triangular Ladder Matrix Algebras

Theorem (B— and Huang 2015)

$M_{\mathcal{L}}$ is closed under matrix multiplication if and only if $\mathcal{L}$ is upper triangular.

Introduced by B— and Huang in 2015.

- Came up in the study of derivations of Lie algebras
- Generalize block upper triangular matrix algebras

Theorem (B— and Huang 2015)

If $\mathcal{L}$ is upper triangular, then $M_{\mathcal{L}}$ is zero product determined under matrix multiplication.

Question

What if matrix multiplication is replaced by the Lie bracket, $[x, y] := xy - yx$?

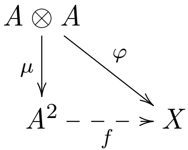
Zero Product Determined Algebras

Introduced by Brešar, Grašič, and Sáncha Ortega in 2009 to further their study of homomorphisms on certain Banach Algebras.

- An *algebra* is a pair (A, μ) where A is a vector space and $\mu : A \otimes A \to A$ is a linear map.
- (A, μ) is *zero product determined* if any $\varphi : A \otimes A \to X$ satisfying

$$\mu(x \otimes y) = 0 \text{ implies } \varphi(x \otimes y) = 0$$

  necessarily factors through μ.



Theorem (Brešar, Grašič, and Ortega 2009)

$M_F^{n\times n}$ is zero product determined under both matrix multiplication and the Lie bracket.

Zero Product Determined Algebras

Theorem (Grašič 2010)

The classical Lie algebras are zero product determined.

Theorem (Wang, Yu, and Chen 2011)

The simple Lie algebras over $\mathbb{C}$ and their parabolic subalgebras are zero product determined.

Theorem (B— and Huang 2015)

(A, μ) *is zero product determined if and only if the kernal of* μ *has a basis consisting of rank-one tensors.*

Theorem (B— unpublished)

The matrix Lie algebra of block form $\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$ *is zero product determined.*

Our Preliminary Results

Theorem

Let $\mathcal{L} = \{(i_1, j_1)\}$. $M_\mathcal{L}$ is zero product determined as a Lie algebra.

If $i_1 < j_1$, then $M_\mathcal{L}$ is abelien, and the conclusion is trivial. Assume $i_1 \geq j_1$. We partition $M_\mathcal{L}$ into blocks of size $a = j_1 - 1$, $b = i_1 - a$, and $c = n - a - b$. We use the rank-nullity theorem to determine that $\operatorname{Ker} \mu$ has dimension

$$a^2b^2 + 2a^2bc + a^2c^2 + 2ab^3 + 4ab^2c + 2abc^2$$
$$-ab - ac + b^4 + 2b^3c + b^2c^2 - b^2 - bc + 1$$

$$M_\mathcal{L} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & \mathfrak{l} & \mathfrak{a} \\ 0 & \mathfrak{h} & \mathfrak{r} \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$M_\mathcal{L} = \mathfrak{h} \ltimes ((\mathfrak{l} \dotplus \mathfrak{r}) \ltimes \mathfrak{a})$$

	$\mathfrak{h}$	$\mathfrak{l}$	$\mathfrak{r}$	$\mathfrak{a}$
$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{l}$	$\mathfrak{r}$	0
$\mathfrak{l}$	$\mathfrak{l}$	0	$\mathfrak{a}$	0
$\mathfrak{r}$	$\mathfrak{r}$	$\mathfrak{a}$	0	0
$\mathfrak{a}$	0	0	0	0

Our Preliminary Results

Taking advantage of the strucure of $M_{\mathcal{L}}$,

$$\begin{array}{ll}
[\mathfrak{h}, \mathfrak{a}] = 0 = [\mathfrak{a}, \mathfrak{h}] & 2ab^2c \\
[\mathfrak{l}, \mathfrak{a}] = 0 = [\mathfrak{a}, \mathfrak{l}] & 2a^2bc \\
[\mathfrak{r}, \mathfrak{a}] = 0 = [\mathfrak{a}, \mathfrak{r}] & 2abc^2 \\
[\mathfrak{a}, \mathfrak{a}] = 0 & a^2c^2 \\
[\mathfrak{l}, \mathfrak{l}] = 0 & a^2b^2 \\
[\mathfrak{r}, \mathfrak{r}] = 0 & b^2c^2 \\
[\mathfrak{h}, \mathfrak{h}] \cong \mathfrak{sl}_b & b^4 - b^2 + 1
\end{array}$$

Still need $2ab^3 + 2ab^2c + 2b^2c - ab - ac - bc$.

Following the method used for $\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$,

$$\begin{array}{ll}
[\mathfrak{h}, \mathfrak{l}] = \mathfrak{l} = [\mathfrak{l}, \mathfrak{h}] & 2ab^3 - ab \\
[\mathfrak{h}, \mathfrak{r}] = \mathfrak{r} = [\mathfrak{r}, \mathfrak{h}] & 2b^3c - bc
\end{array}$$

Still need $2ab^2c - ac$.

Our Preliminary Results

The missing $2ab^2c - ac$ rank-one tensors come from $\mathfrak{l} \otimes \mathfrak{r}$, and $\mathfrak{r} \otimes \mathfrak{l}$, and $(\mathfrak{l} \dot{+} \mathfrak{r}) \otimes (\mathfrak{l} \dot{+} \mathfrak{r})$.

1. $e_{i,a+j} \otimes e_{a+k,a+b+l}$,
   with $1 \leq i \leq a$, $1 \leq j \leq b$, $1 \leq k \leq b$, $1 \leq l \leq c$ and $j \neq k$,
   giving $ab^2c - abc$.
2. Commute the above tensors,
   giving $ab^2c - abc$.
3. $(e_{i,a+j} - e_{i,a+j+1}) \otimes (e_{a+j,a+b+q} + e_{a+j+1,a+b+q})$,
   with $1 \leq i \leq a$, $1 \leq j \leq b - 1$, and $1 \leq q \leq c$,
   giving $abc - ac$.
4. Commute the above tensors,
   giving $abc - ac$.
5. $(e_{i,a+b} + e_{a+b,a+b+q}) \otimes (e_{i,a+b} + e_{a+b,a+b+q})$,
   with $1 \leq i \leq a$ and $1 \leq q \leq c$,
   giving ac.

References

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