Linear Lie Algebras, Block Matrices, and Ladder Matricies

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Linear Lie Algebras

$M^n = M^n(F)$ space of $n \times n$ matrices with entries in F.

M^n is an algebra under matrix multiplication xy.

$\mathfrak{m}^n$ is an algebra under the *Lie bracket* $[x, y]$ (where $[x, y] = xy - yx$).

Upper Triangular Matrices

U^n, set of upper-triangular matrices, is closed under multiplication (and Lie bracket).

$$\begin{pmatrix} * & * & * & * & * & * \\ & * & * & * & * & * \\ & & * & * & * & * \\ & & & * & * & * \\ & & & & * & * \\ & & & & & * \end{pmatrix}$$

$$e_{2,3}e_{3,4} = e_{2,4}$$

$$e_{2,3}e_{3,5} = e_{2,5}$$

$$e_{i,i+k}e_{j,j+l} = \begin{cases} 0 & \text{if } j \neq i+k \\ e_{i,j+l} & \text{if } j = i+k \end{cases}$$

Block Partitions

Multiply block matrices exactly as you multiply regular matrices.

$$\left(\begin{array}{ccc|cc} * & * & * & * & * \\ * & {}_3A_3 & * & {}_3B_2 & * \\ * & * & * & * & * \\ \hline * & {}_2C_3 & * & {}_2D_2 & * \\ * & * & * & * & * \end{array}\right) \left(\begin{array}{ccc|cc} * & * & * & * & * \\ * & {}_3E_3 & * & {}_3F_2 & * \\ * & * & * & * & * \\ \hline * & {}_2G_3 & * & {}_2H_2 & * \\ * & * & * & * & * \end{array}\right)$$

$$= \left(\begin{array}{c|c} {}_3AE_3 + {}_3BG_3 & {}_3AF_2 + {}_3BH_2 \\ \hline {}_2CE_3 + {}_2DG_3 & {}_2CF_2 + {}_2DH_2 \end{array}\right)$$

Block Upper Triangular Matrices

For any partition $\pi = (n_1, n_2, ..., n_k)$ of n, we have the corresponding algebra U_π of block upper triangular matrices, and the Lie algebra $\mathfrak{u}_\pi$.

$$\left(\begin{array}{ccc|cc|c} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ \hline & & & * & * & * \\ & & & * & * & * \\ \hline & & & & & * \end{array}\right) \qquad \left(\begin{array}{c|cc|c|cc} * & * & * & * & * & * \\ \hline & * & * & * & * & * \\ & * & * & * & * & * \\ \hline & & & * & * & * \\ \hline & & & & * & * \\ & & & & * & * \end{array}\right)$$

The $\mathfrak{u}_\pi$ are called the *parabolic subalgebras* of $\mathfrak{m}^n$.

Upper Triangular Ladder Matrices

- A *k-step ladder on n* is a set

$$\mathcal{L} = \{(i_1, j_1), ..., (i_k, j_k)\}$$

with $1 \leq i_1 < i_2 < ... < i_k \leq n$ and
$1 \leq j_1 < j_2 < ... < j_k \leq n$.

- The *ladder matrices on $\mathcal{L}$* is the space

$$M_{\mathcal{L}} := \text{Span}\{e_{i,j} \,;\, (i,j) \in I\} \subseteq M^n$$

where

$$I = \{(i,j) \,;\, \exists (i_t, j_t) \in \mathcal{L}, i \leq i_t, j \geq j_t\}$$

- A ladder $\mathcal{L}$ is called *upper triangular* when $i_t < j_{t+1}$ for $t = 1, 2, ..., k-1$.

Example

Let $\mathcal{L} = \{(3,2), (6,5)\}$,
a 2-step ladder on 6.

$$M_{\mathcal{L}} = \left\{ \left(\begin{array}{c|ccc|cc} 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ \hline 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{array} \right) \right\}$$

Upper Triangular Ladder Matrices

$M_{\mathcal{L}}$ is closed under xy if and only if $\mathcal{L}$ is upper triangular (B— and Huang 2015).

$\mathfrak{m}_{\mathcal{L}}$ is closed under $[x, y]$ whenever $\mathcal{L}$ is upper triangular. But possible also closed for certain non-upper triangular $\mathcal{L}$?

My Actual Research

Structure theory of Lie algebras
(Humphreys 1972, Serre 1965)

Derivations of parabolic Lie algebras
(Leger and Luks 1972, Farnsteiner 1988)

Zero product determined algebras
(Brešar, Grašič, and Ortega 2009, Wang, Yu, and Chen 2011)

Structure of one-step $\mathfrak{m}_{\mathcal{L}}$

$$\mathcal{L} = \{(i_1, j_1)\}$$

$$\left(\begin{array}{cc|cccc} 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|ccc|c} 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ \hline 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Partition n as:

$$n_1 = j_1 - 1$$

$$n_2 = i_1 - n_1$$

$$n_3 = n - n_1 - n_2$$

$$\mathfrak{m}_{\mathcal{L}} = \left(\begin{array}{c|c|c} 0 & \mathfrak{l} & \mathfrak{a} \\ \hline 0 & \mathfrak{h} & \mathfrak{r} \\ \hline 0 & 0 & 0 \end{array}\right)$$

	$\mathfrak{h}$	$\mathfrak{l}$	$\mathfrak{r}$	$\mathfrak{a}$
$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{l}$	$\mathfrak{r}$	0
$\mathfrak{l}$	$\mathfrak{l}$	0	$\mathfrak{a}$	0
$\mathfrak{r}$	$\mathfrak{r}$	$\mathfrak{a}$	0	0
$\mathfrak{a}$	0	0	0	0

$$\mathfrak{m}_{\mathcal{L}} = \mathfrak{h} \ltimes \left((\mathfrak{l} \dotplus \mathfrak{r}) \ltimes \mathfrak{a}\right)$$

Closing Question

$M_{\mathcal{L}}$ is closed under xy if an only if $\mathcal{L}$ is upper triangular. *Which, if any, non-upper triangular* $\mathcal{L}$ *give* $\mathfrak{m}_{\mathcal{L}}$ *closed under* $[x, y]$*?*

Block partition and structural decomposition of $\mathfrak{m}_{\mathcal{L}}$ for one-step $\mathcal{L}$ is complete. *Need block partition scheme and structural decomposition of* $\mathfrak{m}_{\mathcal{L}}$ *for* k*-step* $\mathcal{L}$*.*

$\mathfrak{m}_{\mathcal{L}}$ is zero product determined for one-step $\mathcal{L}$. *Show* $\mathfrak{m}_{\mathcal{L}}$ *is zero product determined for* k*-step* $\mathcal{L}$*.*

References

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