Symmetry Groups

an introduction to group theory through geometry and graph theory

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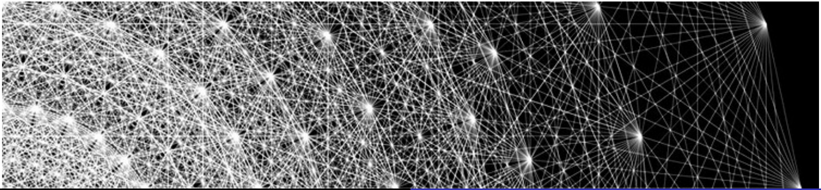

Outline

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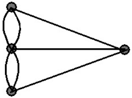

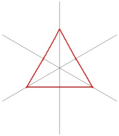

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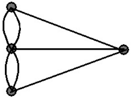

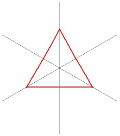

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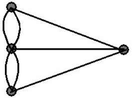

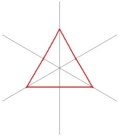

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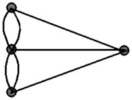

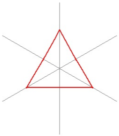

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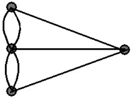

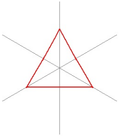

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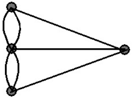

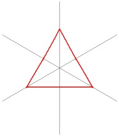

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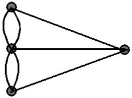

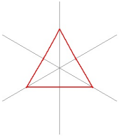

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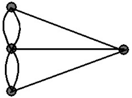

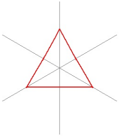

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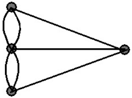

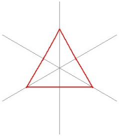

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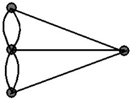

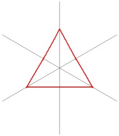

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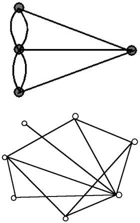

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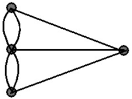

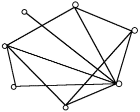

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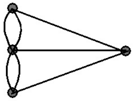

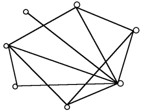

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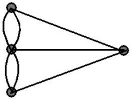

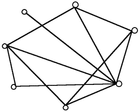

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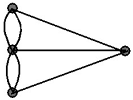

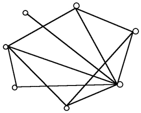

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(Typically, this function is visualized as the adjacency matrix of Γ.)

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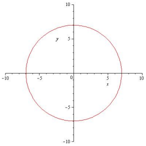
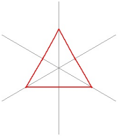
Any subset of $\mathbb{R}^2$ inherits the structure defined on $\mathbb{R}^2$.





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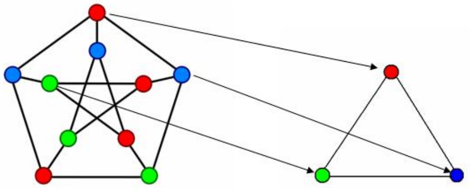

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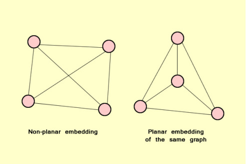

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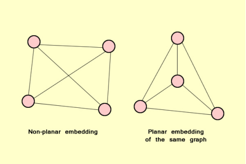

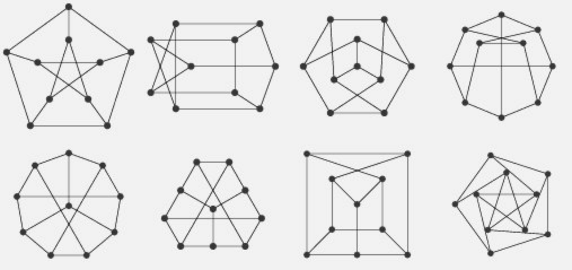

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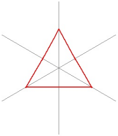

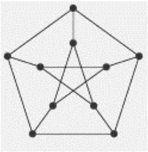

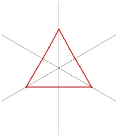

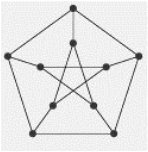

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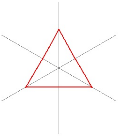

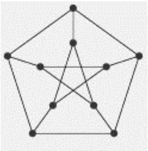

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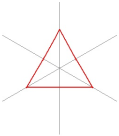

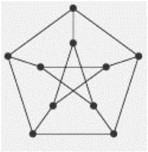

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The same way that numbers can be used to measure various aspects of some object (i.e. number of vertices of a graph, or number or edges, or number of sides of a polygon) automorphism groups are used to measure the symmetry of an object.

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The set $\operatorname{Aut}\mathbb{R}^2$ is also a group, and consists of familiar geometric transformations such:

- Reflecting the entire plane across some line,
- Rotating the entire plane about some point,
- Translating every single point in the plane by some vector.

The same way that numbers can be used to measure various aspects of some object (i.e. number of vertices of a graph, or number or edges, or number of sides of a polygon) automorphism groups are used to measure the symmetry of an object.

The size and complexity of the group $\operatorname{Aut} X$ tells us how symmetric the mathematical object X is.

The invariance theorem

The following theorem says that Automorphism groups are an *invariant* of a graph, ie, a property that isomorphic graphs share.

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Theorem

Suppose Γ *and* Γ' *are graphs. If* Γ *is isomorphic to* Γ', *then* $\mathrm{Aut}\,\Gamma$ *is isomorphic to* $\mathrm{Aut}\,\Gamma'$.

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Exercise: Prove the theorem.

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Theorem

Suppose Γ and Γ' are graphs. If Γ is isomorphic to Γ', then $\operatorname{Aut}\Gamma$ is isomorphic to $\operatorname{Aut}\Gamma'$.

Exercise: Prove the theorem.
Hint: Let f be a graph isomorphism from Γ to Γ'. There's a clever trick you can use that lets f turn elements of $\operatorname{Aut}\Gamma$ into elements of $\operatorname{Aut}\Gamma'$.

The converse of the invariance theorem does not hold. Two non-isomorphic graphs can have isomorphic automorphism groups.

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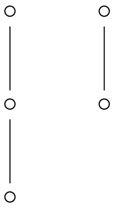
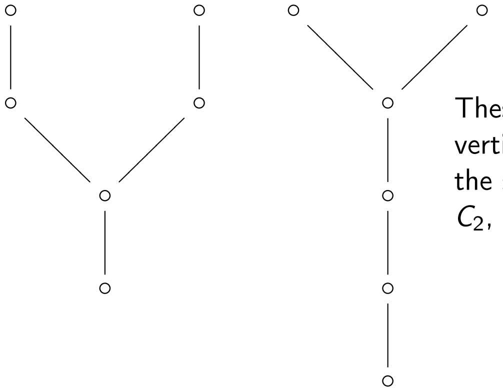
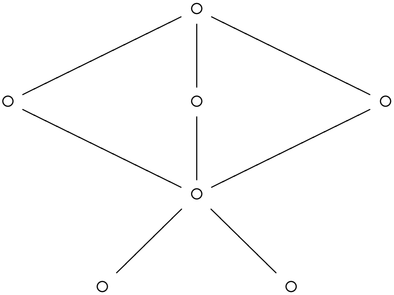
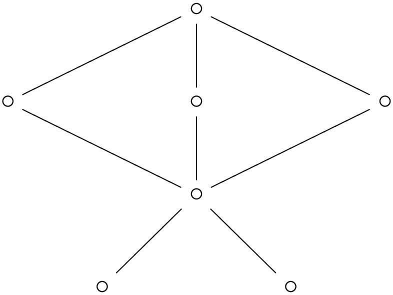


The two graphs to the right have isomorphic automorphism groups (namely, the cyclic group C_2), but are not isomorphic.



These two graphs each have six vertices and five edges, and have the same automorphism group, C_2, but are not isomorphic.





The Automorphism group is $S_3 \times C_2$.

No more yet, maybe someday in the future. I have some more examples in mind, but honestly, it's just really hard to code all these graphs into latex.

- Coxeter, H.S.M. (1969). *Introduction to Geometry,* Second Edition. Wiley Classics Library.
- Hartshorne, R. (2000). *Geometry: Euclid and Beyond.* Springer Undergraduate Texts in Mathematics.
- Weyl, H. (1952). *Symmetry.* Princeton University Press.

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