Parabolic Lie Algebras are Zero Product Determined

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Definitions

Let $(A, *)$ be an algebra over alg-closed, char-0 K.

Definition (preserve zero products)

A bilinear map $\varphi : A \times A \to X$ is said to preserve zero products if

$$\varphi(x, y) = 0 \text{ whenever } x * y = 0.$$

Definition (zero product determined)

A is said to be zero product determined if to each bilinear map $\varphi : A \times A \to X$ that preserves zero products there is a linear map $f : A^2 \to X$ such that

$$\varphi(x, y) = f(x * y).$$

Literature Review

1. Matej Brešar, Mateja Grašič, and Juana Sánchez Ortega. "Zero product determined matrix algebras". In: *Linear Algebra and Its Applications* 430.5 (2009), pp. 1486–1498
2. Mateja Grašič. "Zero product determined classical Lie algebras". In: *Linear and Multilinear Algebra* 58.8 (2010), pp. 1007–1022
3. Dengyin Wang, Xiaoxiang Yu, and Zhengxin Chen. "A class of zero product determined Lie algebras". In: *Journal of Algebra* 331.1 (2011), pp. 145–151

Our Program

Theorem (Wang, Yu, and Chen 2011)

Let $\mathfrak{q}$ be a parabolic subalgebra of a simple Lie algebra $\mathfrak{g}$ over K. $\mathfrak{q}$ is zero product determined.

Can we extend this result:

- to semisimple $\mathfrak{g}$?
- to reductive $\mathfrak{g}$?
- to other constructions on $\mathfrak{q}$?

Our Main Results

Proposition

Let $\mathfrak{q}$ be a parabolic subalgebra of a reductive Lie algebra $\mathfrak{g}$ over K.

1. *$\mathfrak{q}$ is zero product determined.*
2. *$\operatorname{Der}\mathfrak{q}$ is zero product determined.*

Proposition

Let $L(n,k)$ be the Lie algebra of consisting of all $(m+n)\times(m+n)$ matrices of the form

$$\begin{array}{cc} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \end{array}$$

with entries in K. $L(n,k)$ is zero product determined.

Previous Work

Theorem (B and Huang 2015)

Let $(A, *)$ be an algebra.

1. *A is ZPD iff $\operatorname{Ker} *$ is generated by rank-one tensors in $A \otimes A$.*
2. *Abelian Lie algebras are ZPD.*
3. *A direct sum is ZPD iff each summand is ZPD.*

Theorem (B and Huang manuscript pending submission)

Let $\mathfrak{q}$ be a parabolic subalgebra of a reductive Lie algebra $\mathfrak{g}$ over K or $\mathbb{R}$

1. *$\mathfrak{q}$ decomposes as $\mathfrak{q} = \mathfrak{g}_Z \dot{+} \mathfrak{c} \dot{+} [\mathfrak{q}, \mathfrak{q}]$*
2. *$\mathfrak{L} = \{$deriv. $D | D(\mathfrak{q}) \subseteq \mathfrak{g}_Z, D([\mathfrak{q}, \mathfrak{q}]) = 0\}$ is an ideal of $\operatorname{Der} \mathfrak{q}$.*
3. *$\operatorname{Der} \mathfrak{q} \cong \mathfrak{L} \oplus \operatorname{ad} \mathfrak{q}$.*

Proof of Main Results

- $\mathfrak{q}$ is ZPD

  Write $\mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S$.

  $\mathfrak{q}_S$ is ZPD by Wang, Yu, and Chen 2011 and direct sums.

  $\mathfrak{g}_Z$ is ZPD since it is abelian.

  $\mathfrak{q}$ is ZPD by direct sums.

- $\operatorname{Der}\mathfrak{q}$ is ZPD

  $\operatorname{Der}\mathfrak{q} \cong \mathfrak{L} \oplus \operatorname{ad}\mathfrak{q}$.

  $\operatorname{ad}\mathfrak{q}$ is ZPD by Wang, Yu, and Chen 2011 and direct sums.

  $\mathfrak{L} \cong L(n,k)$, ZPD by the next proposition.

  $\operatorname{Der}\mathfrak{q}$ is ZPD by direct sums.

Proof of Main Results

- $L(n,k)$ is ZPD
  - Dimension-counting shows

$$\dim \operatorname{Ker} * = n^4 + 2n^3k + n^2 + n^2k^2 - n^2 - nk + 1$$

  - $n^4 - n^2 + 1$ from upper-right block, since it is ZPD
  - n^2k^2 from upper-left block, since it is abelian
  - $2n^3k - 2n^2k$ of the form

$$e_{i,j} \otimes e_{l,n+q}, \quad e_{l,n+q} \otimes e_{i,j} \quad : \quad i,j,k \leq n, \quad q \leq k, \quad j \neq l$$

  - $2n^2k - 2nk$ of the form

$$(e_{i,j} - e_{i,j+1}) \otimes (e_{j,n+q} + e_{j+1,n+q}),$$

$$(e_{j,n+q} + e_{j+1,n+q}) \otimes (e_{i,j} - e_{i,j+1}) \quad : \quad i \leq n, \quad j \leq n-1, \quad q \leq k$$

  - nk of the form

$$(e_{i,i} + e_{i,n+q}) \otimes (e_{i,i} + e_{i,n+q}) \quad : \quad i \leq n, \quad q \leq k$$