Derivations of Parabolic Lie Algebras with Applications to Zero Product Determined Algebras

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Derivations

Let $\mathfrak{g}$ be a Lie algebra.

Definition

A *derivation* of $\mathfrak{g}$ is a linear map $D : \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying

$$D\big([x,y]\big) = \big[D(x), y\big] + \big[x, D(y)\big]$$

for all $x, y \in \mathfrak{g}$.

$\mathrm{Der}\,\mathfrak{g}$ denotes the set of all derivations of $\mathfrak{g}$.
It is a Lie algebra under $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$.

Question

Can we characterize all of the derivations of $\mathfrak{g}$?

Derivations

Definition

Derivations of the form $[x, \cdot]$ for $x \in \mathfrak{g}$ are called *inner* derivations. The set of inner derivation of $\mathfrak{g}$ is denoted $\operatorname{ad} \mathfrak{g}$. All other derivations are termed *outer* derivations.

Theorem (classical result [1, 7])

If $\mathfrak{g}$ is semisimple over a field of characteristic not 2, then $\operatorname{Der} \mathfrak{g} = \operatorname{ad} \mathfrak{g}$.

Further Results

- $\mathfrak{q}$ is a parabolic subalgebra of semisimple $\mathfrak{g}$ over a $\mathbb{C}$-like field. $\operatorname{Der} \mathfrak{q} = \operatorname{ad} \mathfrak{q}$ (independently by Legar and Luks and by Tolpygo — 1972 [6, 8]).
- Outer derivation of Kac-Moody algebras and their Borel subalgebras characterized (Farnsteiner — 1988 [4]).

Zero Product Determined Algebras

Let $\mathcal{A}$ be an algebra with multiplication $*$.

Definition

$\mathcal{A}$ is called *zero product determined* (ZPD) if for each bilinear map $\varphi : \mathcal{A} \times \mathcal{A} \longrightarrow V$, if

$$\varphi(x, y) = 0 \text{ whenever } x * y = 0,$$

then there is a linear map $f : \mathcal{A}^2 \longrightarrow V$ satisfying

$$\varphi(x, y) = f(x * y)$$

for all $x, y \in \mathcal{A}$.

Question

Which Lie algebras are ZPD?

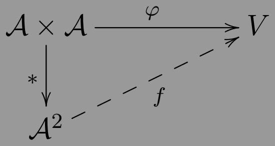

Figure : $\mathfrak{g}$ is ZPD

Zero Product Determined Algebras

Theorem (B, Huang — 2014 [3])

Let $(\mathcal{A}, *)$ be an algebra. Denote the map $x \otimes y \mapsto x * y$ by $\mu : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}^2$. $\mathcal{A}$ is ZPD if and only if $\operatorname{Ker} \mu$ is generated by elementary tensors.

Further Results

- $\mathfrak{sl}(\mathbb{F}^n)$ and $\mathfrak{gl}(\mathbb{F}^n)$ are ZPD (Brešar, Grašič, and Ortega — 2009 [2]).
- The classical families B_r, C_r, and D_r are ZPD (Grašič — 2010 [5]).
- Parabolic subalgebras of simple Lie algebras over a $\mathbb{C}$-like field are ZPD (Wang, et al. — 2011 [9]).
- Direct sums of ZPD algebras are ZPD (B, Huang — 2014 [3]).

Results

Theorem

Let $\mathfrak{q}$ be a parabolic subalgebra of a reductive Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ or over a $\mathbb{C}$-like field. Let $\mathfrak{L}$ be the set of all linear transformations mapping $\mathfrak{q}$ into $\mathfrak{q}_Z$ that send $[\mathfrak{q}, \mathfrak{q}]$ to 0. Then $\operatorname{Der}\mathfrak{q}$ has the Lie algebra direct sum decomposition

$$\operatorname{Der}\mathfrak{q} = \mathfrak{L} \oplus \operatorname{ad}\mathfrak{q}.$$

This helps us answer our original question: whether or not $\operatorname{Der}\mathfrak{q}$ is ZPD.

Question

Can we describe the members of $\mathfrak{L}$ more concretely?

Results

$$\mathfrak{L} = \left\{ f : \mathfrak{q} \xrightarrow{\text{linear}} \mathfrak{q} \;\middle|\; f(\mathfrak{q}) \subseteq \mathfrak{q}_Z \text{ and } f\big([\mathfrak{q},\mathfrak{q}]\big) = 0 \right\}.$$

We use the *Langland's decomposition* of $\mathfrak{q}$ to understand members of $\mathfrak{L}$.
Write $\mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$, then $\mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S$ and $\mathfrak{q}_Z = \mathfrak{g}_Z$.
$\mathfrak{q}_S$ decomposes as $\mathfrak{q}_S = \mathfrak{l} \ltimes \mathfrak{n}$, with $\mathfrak{n}$ nilpotent and $\mathfrak{l}$ reductive, so $\mathfrak{l} = \mathfrak{l}_Z \oplus \mathfrak{l}_S$.

We show that $[\mathfrak{q},\mathfrak{q}] = \mathfrak{l}_S + \mathfrak{n}$, so that

$$\begin{aligned} \mathfrak{q} &= \mathfrak{g}_Z + \mathfrak{l} + \mathfrak{n} \\ &= \mathfrak{g}_Z + \mathfrak{l}_Z + \mathfrak{l}_S + \mathfrak{n} \\ &= \mathfrak{g}_Z + \mathfrak{l}_Z + [\mathfrak{q},\mathfrak{q}]. \end{aligned}$$

Then members of $\mathfrak{L}$ are realized as matrices with block form

$$\begin{array}{c c} & \begin{array}{ccc} \mathfrak{g}_Z & \mathfrak{l}_Z & [\mathfrak{q},\mathfrak{q}] \end{array} \\ \begin{array}{c} \mathfrak{g}_Z \\ \mathfrak{l}_Z \\ [\mathfrak{q},\mathfrak{q}] \end{array} & \begin{pmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{array}$$

Results

Theorem (Wang, et al. — 2011 [9])

Let $\mathfrak{q}$ be a parabolic subalgebra of a simple Lie algebra $\mathfrak{g}$ over a $\mathbb{C}$-like field. $\mathfrak{q}$ is ZPD.

Wang, et al.'s proof has a minor error, which we correct.

Lemma

An abelian Lie algebra is ZPD.

Proof.

The lemma follows from the elementary tensor description of ZPD from [3]. □

Theorem

If $\mathfrak{g}$ is reductive, then $\mathfrak{q}$ is ZPD.

Proof.

$\mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S$. $\mathfrak{q}_S$ is ZPD by the result of Wang, et al. in [9], and by direct sum decomposition, $\mathfrak{q}$ is ZPD. □

Results

Theorem

Let $\mathfrak{q}$ be a parabolic subalgebra of a reductive Lie algebra $\mathfrak{g}$ over a $\mathbb{C}$-like field. Then the derivation algebra $\operatorname{Der}\mathfrak{q}$ is ZPD.

Proof.

From earlier, $\operatorname{Der}\mathfrak{q} = \mathfrak{L} \oplus \operatorname{ad}\mathfrak{q}$.
$\operatorname{ad}\mathfrak{q} \cong \mathfrak{q}_S$ is ZPD by the result of Wang, et al. in [9].
We show that $\mathfrak{L}$ is ZPD with a tensor argument.
Then $\operatorname{Der}\mathfrak{q}$ is ZPD by direct sum decomposition. □

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