Continuous Symmetry Groups

a very gentle introduction
to the beautifully geometric
theory of Lie groups

Daniel Brice

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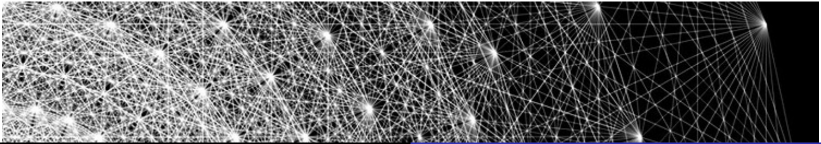

Outline

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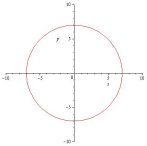

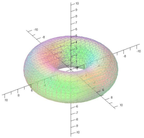

- Symmetries in Geometry

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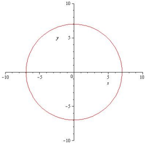

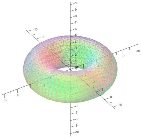

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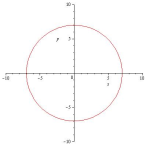

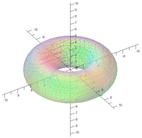

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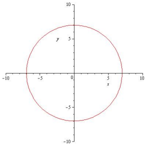

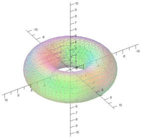

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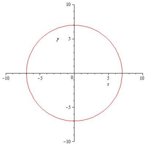

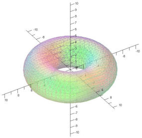

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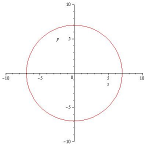

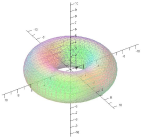

- Symmetries in Geometry
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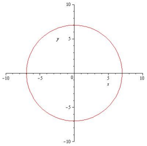

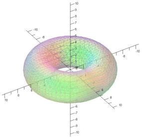

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- One-Parameter Groups

Geometry is concerned with *figures* and *isometries*.

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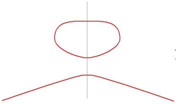
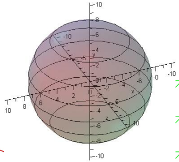
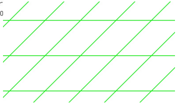
A *figure* $\mathcal{F}$ is a subset of $\mathbb{R}^n$.







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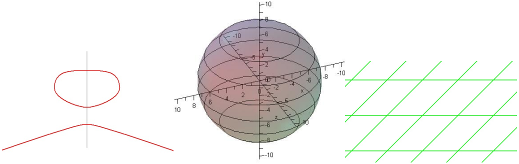

An *isometry* is a function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ that is:

1. one-to-one and onto,
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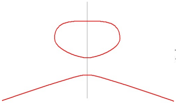

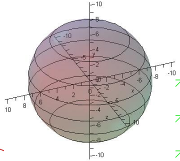

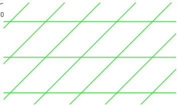

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A *symmetry* ψ of a figure $\mathcal{F}$ is an isometry that leaves $\mathcal{F}$ *invariant*, meaning that $\psi(\mathcal{F}) = \mathcal{F}$.

For any figure $\mathcal{F}$, the set of symmetries of $\mathcal{F}$ forms a group.

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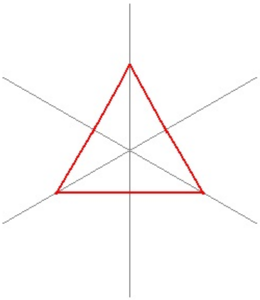

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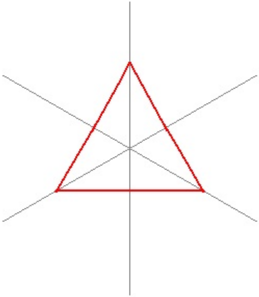

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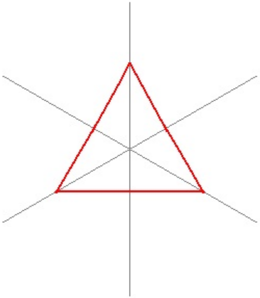

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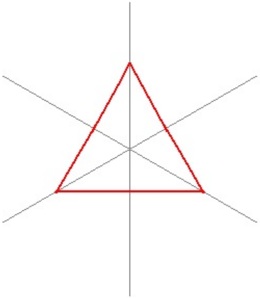

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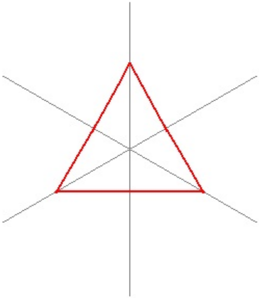

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We'll call this group **Sym**$(\mathcal{F})$.
We will also consider subgroups of **Sym**$(\mathcal{F})$.

$\mathcal{F}$

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$\textbf{Sym}(\mathcal{F}) = C_3$

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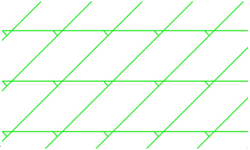

$\mathcal{H}$

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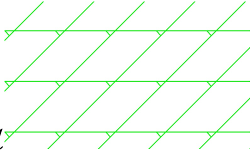

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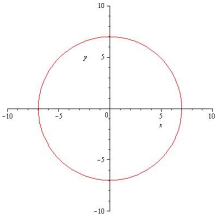

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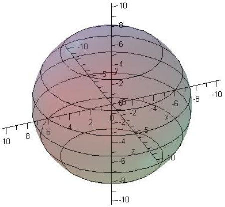

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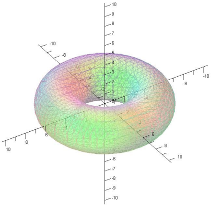

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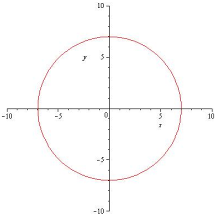

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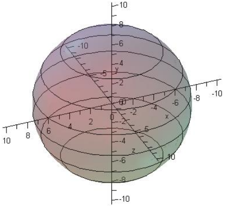

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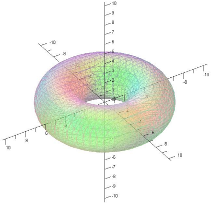

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$\textbf{Sym}(\mathcal{F}) = S^1 \cup \{\text{reflections that fix the origin}\}$

Symmetries in Geometry
Figures with Continuous Symmetry Groups

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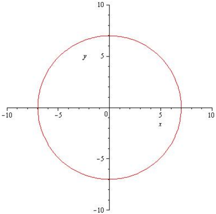

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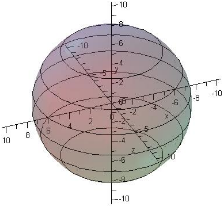

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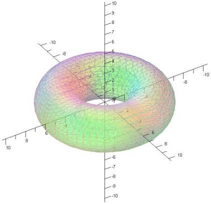

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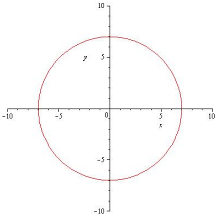

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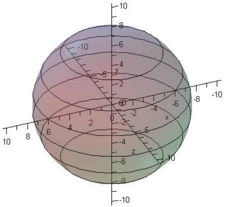

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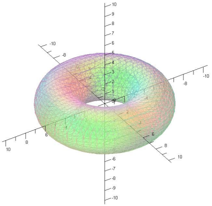

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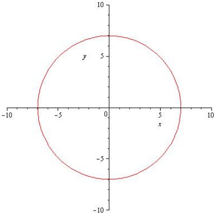

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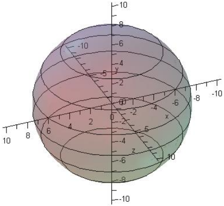

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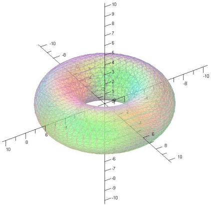

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$\mathbf{Sym}(\mathcal{H})$ is today's challenge question.

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The group operation $*$ is matrix multiplication.

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Unit Circle Representation ($\mathbb{R}$ modulo 2π):
$S^1 = \{[x] : x \in \mathbb{R}, [x_1] = [x_2] \text{ if and only if } x_1 \equiv x_2 \pmod{2\pi}\}$

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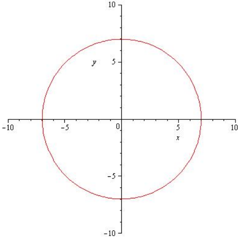

Two Representations of S^1

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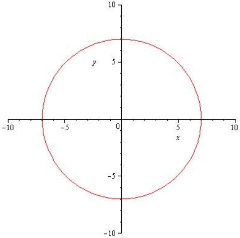

The group operation $*$ is addition of equivalence classes via reduction modulo 2π.

$$[x_1] * [x_2] = [x_1 + x_2]$$

Matrix Representation: $S^1 = \{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R}\}$.
The group operation $*$ is matrix multiplication.

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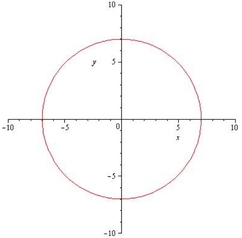

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$[x_1] * [x_2] = [x_1 + x_2]$
$g : \mathbb{R} \to S^1$ by $g(x) = x \mod 2\pi$

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2. the intersection of finitely many open sets is open, and
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Two topological spaces X and Y are called *homeomorphic* when there is a function between them that is one-to-one, onto, continuous, and whose inverse is also continuous.

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In other words, group multiplication and group inversion are continuous function in the topology sense.

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A cyclic group can be thought of as a pair (C, f) where C is a group and f an onto group homomorphism $f : \mathbb{Z} \to C$.

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A *one-parameter group* is a pair (G, f) where G is a non-trivial continuous group and $f : \mathbb{R} \to G$

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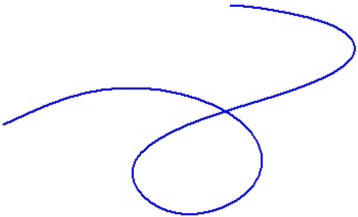
In a sense, we can think of $t \in \mathbb{R}$ as the time parameter of a path that traces out our group G, visiting each point of G at least once.



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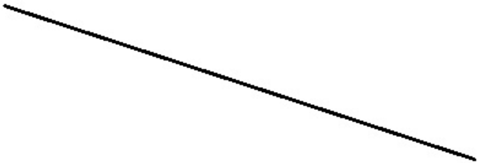
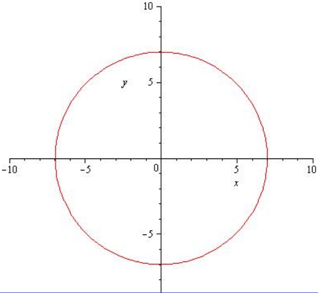
So there are only two essentially different one-parameter groups.





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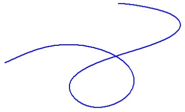

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- h and h^{-1} are continuous.
- Therefore, h is an isomorphism of groups and a homeomorphism of topological spaces. QED.

Further Reading
Discrete Symmetry Groups

- Coxeter, H.S.M. (1969). *Introduction to Geometry,* Second Edition. Wiley Classics Library.
- Hartshorne, R. (2000). *Geometry: Euclid and Beyond.* Springer Undergraduate Texts in Mathematics.
- Weyl, H. (1952). *Symmetry.* Princeton University Press.

Discrete symmetry groups have applications in chemistry, quantum theory, and graphic design.

Continuous Symmetry Groups

What we have called *continuous groups* are more often called *topological groups* in the literature. The most important class of topological groups is the class of *Lie groups*, which are not only continuous, but differentiable.

- Bourbaki, N. (1980). *Lie Groups and Lie Algebras,* Chapter 1-3. Springer-Verlang.
- Munkres, J.R. (2000). *Topology,* Second Edition. Prentice Hall.
- Stillwell, J. (2008). *Naive Lie Theory*. Springer Undergraduate Texts in Mathematics.

Topological groups have applications in relativity, string theory, and computer graphics.

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