Chapter 1 Exercises

Exercise 1.4

1. assertion
2. assertion
3. not assertion
4. assertion
5. assertion
6. assertion
7. assertion
8. not assertion
9. assertion

Exercises 1.13

Which of the following is possible? If it is possible, give an example. If it is not possible, explain why.

1. A valid deduction that has one false hypothesis and one true hypothesis.

   POSSIBLE

   H1. Paris is in France (true)
   H2. Big Ben in is Paris (false)
   Conc. Big Ben is in France (false)
   (valid)

2. A valid deduction that has a false conclusion.

   POSSIBLE

   Same deduction above works.

3. A valid deduction that has at least one false hypothesis, and a true conclusion.

   POSSIBLE

   H1. If this homework is hard, then logic is hard. (true)
   H2. This homework is hard (false)
   Conc. Logic is hard. (true)
   (valid)

   (Logic is indeed hard, but for reasons other than those found in this homework assignment. This homework is pretty straightforward.)

4. A valid deduction that has all true hypotheses, and a false conclusion.

   IMPOSSIBLE

   In order for a deduction to be valid, the conclusion must be true whenever the hypotheses are true.

5. An invalid deduction that has at least one false hypothesis, and a true conclusion.

   POSSIBLE

   H1. Either I'll have a soup or a salad. (true)
   H2. I'll have a salad. (false)
   Conc. I'll have a soup. (true)
   (invalid)

   (This has the form "Either A or B; B; therefor A". This type of reasoning is invalid, whether or not A is true.)

Exercise 1.14

H1. The robber(s) left in a truck
H2. No one other than A, B, and C was involved.
H3. C never commits a crime without inviting A.
H4. B doesn't know how to drive.

Was A involved?

From H1, H2, and H4, A or C must have driven, so at least one of them was involved. In case it were A who drove, then we're done---he's involved. On the other hand, if it were C who drove, then H3 says that C would have invited A, so A at least knew about the heist beforehand, which we can go ahead and count that as "involved."

Exercise 1.15

1.

If Alice is a Knight, then so is Bob, so in that case, Bob is reliable. But then Bob says that Alice is a Knave, which contradicts what Alice said about herself. Since assuming that Alice is a Knight leads to a contradiction, Alice must not be a Knight, so Alice is definitely a Knave.

Since Alice is a Knave, we know that Bob was telling the truth about her, so Bob must be a Knight.

2.

If Charlie were a Knave, then his statement "not all of us are Knights" would be false, so all of them would be Knights, including Charlie. Since this is impossible, Charlie can't be a Knave, and so must be a Knight.

Since Charlie is a Knight, his statement must be true, so at least one of Diane or Ed is a Knave. Diane's statement, "not all of us are Knaves," is already known to us to be true, since Charlie is a Knight, so since Diane just told the truth, so must be a Knight. That makes Ed a Knave.

3.

If George were a telling the truth, then Frances would have just claimed to be a Knave. But no one can ever claim to be a Knave (think about it), so George is a lying sack of Knave.

George also said that Frances is a Knave. Since we know George always lies, Frances must be a Knight.

Exercise 1.16

```
4 1 3 2
2 3 1 4
1 4 2 3
3 2 4 1

1 2 3 4
3 4 1 2
2 1 4 3
4 3 2 1

1 4 2 3
3 2 4 1
4 3 1 2
2 1 3 4
```

Exercise 1.17

1.

I claim that the secret code is 3521. To verify, we check that it has
the right number of "bulls" and "cows" with the previous guesses.

My Code	Guess	"Bulls"	"Cows"	Works?
3521	1234	0	3	yes
3521	2354	0	3	yes
3521	3642	1	1	yes
3521	5143	0	3	yes
3521	4512	1	2	yes

The above computations verify that my code is at least _consistent_ with the answers to the previous guesses. But the textbook author tells us that these guesses give us enough to find the code, so mine must be it.

2.

I claim that the secret code is 4155. We verify below:

My Code	Guess	"Bulls"	"Cows"	Works?
4155	1234	0	2	yes
4155	4516	1	2	yes
4155	4621	1	1	yes
4155	6543	0	2	yes
4155	5411	0	3	yes

So, my code is consistent, and again the textbook author tells us there's only one, so it must be the one I provided.

Chapter 2 Part 1

Exercises 2.13, 2.14, 2.21, 2.25, 2.34, 2.36, 2.39, 2.40

Exercise 2.13

* G: Gregor plays first base.
* L: The team will lose.
* E: Evan plays first base.
* M: There will be a miracle.
* C: Gregor's mom will bake cookies.

1. G => L
2. (G v E) => -M
3. -(G v E) => M
4. -M => L
5. M => -C

Exercise 2.14

1.

  * M: Dorothy plays the piano in the morning.
  * C: Roger wakes up cranky.
  * D: Dorothy is distracted.

  H1: M => C
  H2: -D => M
  Conc: -C => D

2.

  * R: It will rain on Tuesday.
  * S: It will snow on Tuesday.
  * N: Neville will be sad.
  * C: Neville will be cold.

  H1: R v S
  H2: R => N
  H3: S => C
  Conc: N v C

3.

  * Z: Zoog remembered to do his chores.
  * C: Things are clean.
  * N: Things are neat.

  H1: Z => (C & -N)
  H2: -Z => (N & -C)

```
  Conc: (N v C) & -(N & C)
```

Exercise 2.21

```
1. (A v C) => -(A => B)
   a. A = T, B = F, C = F
      (T v F) => -(T v F)
      T => -(T)
      F
   b. A = F, B = T, C = F
      (F v F) => -(F => T)
      F => -T
      F => F
      T
2. (P v -(Q => R)) => ((P v Q) & R)
   a. P = Q = R = T
      (T v -(T => T)) => ((T v T) & T)
      (T v -T) => (T & T)
      T => T
      T
   b. P = T, Q = F, R = T
      (T v -(F => T)) => ((T v F) & T)
      (T v -T) => (T & T)
      T => T
      T
   c. P = F, Q = T, R = T
      (F v -(T => T)) => ((F v T) & T)
      (F v -T) => (T & T)
      (F v F) => (T & T)
      F => T
      T
   d. P = Q = R  =F
      (F v -(F => F)) => ((F v F) & F)
      (F v -T) => (F & F)
      (F v F) => (F & F)
      F => F
      T
3. ((U & -V) v (V & -W) v (W & -U)) => -(U & V & W)
   a. U = V = W = T
      ((T & -T) v (T & -T) v (T & -T)) => -(T & T & T)
      ((T & F) v (T & F) v (T & F)) => -(T & T & T)
      (F v F v F) => -T
      F => F
      T
   b. U = T, V = T, W = F
      ((T & -T) v (T & -F) v (F & -T)) => -(T & T & F)
      ((T & F) v (T & T) v (F & F)) => -F
      (F v T v F) => T
      T => T
      T
```

   c. U = F, V = T, W = F
      ((F & –T) v (T & –F) v (F & –F)) => –(F & T & F)
      ((F & F) v (T & T) v (F & T)) => –F
      (F v T v F) => T
      T => T
      T
   d. U = V = W = F
      ((F & –F) v (F & –F) v (F & –F)) => –(F & F & F)
      ((F & T) v (F & T) v (F & T)) => –F
      (F v F v F) => T
      F => T
      T
4. (X v –Y) & (X => Y)
   a. X = Y = T
      (T v –T) & (T => T)
      (T v F) & (T => T)
      T & T
      T
   b. X = T, Y = F
      (T v –F) & (T => F)
      (T v T) & (T => F)
      T & F
      F
   c. X = F, Y = T
      (F v –T) & (F => T)
      (F v F) & (F => T)
      F & T
      F
   d. X = Y = F
      (F v –F) & (F => F)
      (F v T) & (F => F)
      T & F
      F

Exercises 2.25

1. A => (A & B) is not true when A = T and B = F, since
   T => (T & F)
   T => F
   F
   A => (A & B) is true when A = F and B = T, since
   F => (F & T)
   F => F
   F
2. (A v B) => A is true when A = T and B = T, since
   (T v T) => T
   T => T
   T
   (A v B) => A is false when A = F and B = T, since
   (F v T) => F

T => F
F

3. (A <=> B) v (A & -B) is true when A = T and B = T, since
   (T <=> T) v (T & -T)
   T v F
   T
   (A <=> B) v (A & -B) is false when A = F and B = T, since
   (F <=> T) v (F & -T)
   (F <=> T) v (F & F)
   F v F
   F
4. (P & -(Q & R)) v (Q => R) is false when P = F, Q = T, and R = F,
   (F & -(T & F)) v (T => F)
   (F & -F) v (T => F)
   (F & T) v (T => F)
   F v F
   F
   (P & -(Q & R)) v (Q => R) is true when P = Q = R = T,
   (T & -(T & T)) v (T => T)
   (T & -T) v (T => T)
   (T & F) v (T => T)
   F v T
   T
5. (X => Z) => (Y => Z) is false when X = F, Y = T, and Z = F,
   (F => F) => (T => F)
   T => F
   F
   (X => Z) => (Y => Z) is true when X = T, Y = T, and Z = T,
   (T => T) => (T => T)
   T => T
   T

Exercise 2.34

1. A v B v -C, (A v B) & (C => A)

A	B	C	A v B v -C	(A v B) & (C => A)
T	T	T	T	T
T	T	F	T	T
T	F	T	T	F
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	F
F	F	F	T	F

They are not logically equivalent because they disagree in the third and the eight rows.

2. (P => Q) v (Q => P), P v Q

P	Q	(P => Q) v (Q => P)	P v Q
T	T	T	T
T	F	T	T
F	T	T	T
F	F	T	F

They are not logically equivalent because they disagree in the fourth row.

3. (X & Y) => Z, X v (Y => Z)

X	Y	Z	(X & Y) => Z	X v (Y => Z)
T	T	T	T	T
T	T	F	F	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	T

They are not logically equivalent because they disagree in the second and the seventh rows.

Exercise 2.36

1. Rules of negation ("De Morgan's Laws")

--A == A

A	--A
T	T
F	F

-(A & B) == -A v -B

A	B	-(A & B)	-A v -B
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

-(A v B) == -A & -B

A	B	-(A v B)	-A & -B
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

-(A => B) == A & -B

A	B	-(A => B)	A & -B
T	T	F	F
T	F	T	T
F	T	F	F
F	F	F	F

-(A <=> B) == A <=> -B

A	B	-(A <=> B)	A <=> -B
T	T	F	F
T	F	T	T
F	T	T	T
F	F	F	F

2. Commutativity of &, v, and <=>

   A & B == B & A

A	B	A & B	B & A
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	F

A v B == B v A

A	B	A v B	B v A
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

A <=> B == B <=> A

A	B	A <=> B	B <=> A
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

3. Associativity of & and v

   (A & B) & C == A & (B & C)

A	B	C	A & B	B & C	(A & B) & C	A & (B & C)
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

   (A v B) v C == A v (B v C)

A	B	C	A v B	B v C	(A v B) v C	A v (B v C)
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	F	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

Exercise 2.39

1. -((A v B) => (C & D))
   == (A v B) & -(C & D)
   == (A v B) & (-C v -D)

2. -((A => B) v (C & D))
   == -(A => B) & -(C & D)
   == A & -B & (-C v -D)

3. -(A => (B => (C => D)))
   == A & -(B => (C => D))
   == A & (B & -(C => D))
   == A & (B & (C & - D))
   == A & B & C & -D

4. -(((A => B) => C) => D)
   == ((A => B) => C) & -D

5. -((P v -Q) & R)
   == -(P v -Q) v -R
   == (-P & Q) v -R

6. -(P & Q & R & S)
   == -((P & Q) & (R & S))
   == -(P & Q) v -(R & S)
   == (-P v -Q) v (-R v -S)
   == -P v -Q v -R v -S

7. -((P => (Q & -R)) v (P & -Q))
   == -(P => (Q & -R)) & -(P & -Q)
   == (P & -(Q & -R)) & (-P v Q)
   == (P & (-Q v R)) & (-P v Q)
   == P & (-Q v R) & (-P v Q)

Exercise 2.40

1. It's raining, and the bus is on time.

2. Either I'm not sick or I'm not tired.

3. -(P v (Q & R))
   == -P & -(Q & R)
   == -P & (-Q v -R)
   The Pope isn't here, and either the Queen isn't here or the Russian
\
   isn't here.

4. -[(T => (I v O)) & (-B v A)]
   == -(T => (I v O)) v -(-B v A)
   == (T & -(I v O)) v (B & -A)
   == (T & -I & -O) v (B & -A)
   Either Tom forgot his backpack and Sam ate neither a pickle nor a
   potato or Bob will have lunch and Alice won't drive to the store.

Chapter 2 Part 2

Exercises 2.41, 2.42, 2.43, 2.45, 2.47, 2.49, 2.52, 2.53

Exercise 2.41

Show that A => B is not logically equivalent to its converse B => A.

We will show this with a truth table.

A	B	A => B	B => A
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

They disagree in rows 2 and 3, so they are not logically equivalent.

Exercise 2.42

Show that A => B is not logically equivalent to its inverse –A => –B.

Again, we'll use a truth table.

A	B	A => B	–A	–B	–A => –B
T	T	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

They disagree in rows 2 and 3, so they are not logically equivalent.

Exercise 2.43

Show that A => B is logically equivalent to its contrapositive –B => –A.

No surprise, we'll use a truth table.

A	B	A => B	–A	–B	–B => –A
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

In every row they agree, so they are logically equivalent.

Exercise 2.45

State (a) the converse and (b) the contrapositive of each implication.

1. If the students comes to class, then the teacher lectures.

   Converse: If the teacher lectures, then the students come to class.

   Cntrpstv: If the teacher didn't lecture, then the students didn't come to class.
   (or)
   If the teacher doesn't lecture, then the students don't come to class.
   (Tense is hard to model in propositional logic, so either of the above might be the contrapositive. A similar comment applies to all of the following problems.)

2. If it rains, then I carry my umbrella.

   Converse: If I carry my umbrella, then it rains.

   Cntrpstv: If I didn't carry my umbrella, then it didn't rain.

3. If I have to go to school this morning, then today is a weekday.

   Converse: If today is a weekday, then I have to go to school this morning.

   Cntrpstv: If today isn't a weekday, then I don't have to go to school this morning.

4. If you give me $5, I can take you to the airport.

   Converse: If I can take you to the airport, then you give me $5.

   Cntrpstv: If I can't take you to the airport, then you won't give me $5.

5. If the Mighty Ducks are the best hockey team, then pigs can fly.

   Converse: If pigs can fly, then the Mighty Ducks are the best hockey team.

   Cntrpstv: If pigs can't fly, then the Mighty Ducks aren't the best hockey team.

6. Alberta is a province.

Converse: Alberta is a province.

Cntrpstv: Alberta is a province.

The assertion is merely a propositional atom, i.e., it's of the form P. If it were P => Q (or P v Q or P & Q) then taking converse or contrapositive would be meaningful, but since it's only an atom, converse and contrapositive leave it the same. (Think about how taking the reciprocal of 1 doesn't do anything, or taking the negative of 0 doesn't do anything, in the realm of numbers. Similar concept.)

7. If (you want will do well in your math class), then (you need to do all of the homework problems).

   Converse: If you need to do all of the homework problems, then you want will do well in your math class.

   Cntrpstv: If you do not need to do all of your homework problems, then you do not want will do well in your math class.

   (I think there's a typo in the book <.<)

Exercise 2.47

Answer each of the questions below and justify your answer.

1. Suppose (_A_ & _B_) => _C_ is neither a tautology nor a contradiction. What can you say about the deduction "_A_, _B_, .: _C_"?

   Well, if (_A_ & _B_) => _C_ isn't a tautology, then there's a valuation in which (_A_ & _B_) is true but _C_ is not. In that valuation, the hypotheses of "_A_, _B_, .: _C_" would be true but the conclusion would be false, so that deduction is invalid.

2. Suppose _A_ is a contradiction. What can you say about the deduction "_A_, _B_, .: _C_"?

   If _A_ is a contradiction, then it's false in every valuation, so there are zero valuation in which the hypotheses are true. In each of those valuations (i.e., all zero of them) the conclusion happens to be true as well, so the deduction is valid (although we can all agree that it's valid for silly reasons).

3. Suppose that _C_ is a tautology. What can you say about the deduction
"_A_, _B_ .: _C_"?

If the conclusion _C_ is always true, then it's true whenever the hypotheses are true, so the deduction is valid (again, for silly reasons).

Exercise 2.49

We will use truth tables to show that these are valid deductions, but to save on space, we will only show valuations (rows) in which all the hypotheses are true (as the directions suggest).

1. repeat: A, .: A

```
A | A
--+--
T | T
```

2. &-intro: A, B, .: A & B

```
A | B | A & B
--+---+------
T | T | T
```

3. &-elim: A & B, .: A

```
A & B | A
------+--
T     | T
```

&-elim: A & B, .: B

```
A & B | B
------+--
T     | T
```

4. v-intro: A, .: A v B

```
A | A v B
--+------
T | T
```

v-intro: B, .: A v B

```
B | A v B
--+------
T | T
```

5. v-elim: A v B, -A, .: B

A	B	A v B	-A	B
F	T	T	T	T

6. =>-elim: A => B, A. .: B

A	B	A => B	A	B
T	T	T	T	T
F	T	T	T	T

7. <=>-intro: A => B, B => A, .: A <=> B

A	B	A => B	B => A	A <=> B
T	T	T	T	T
F	F	T	T	T

8. <=>-elim: A <=> B, .: A => B

A	B	A <=> B	A => B
T	T	T	T
F	F	T	T

<=>-elim: A <=> B, .: B => A

A	B	A <=> B	B => A
T	T	T	T
F	F	T	T

9. proof by cases: A v B, A => C, B => C, .: C

(this one is a little confusing, so we'll fill out the entire truth table)

A	B	C	A v B	A => C	B => C	C
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	T	T	T	T
T	F	F	T	F	T	F
F	T	T	T	T	T	T
F	T	F	T	T	F	F
F	F	T	F	T	T	T
F	F	F	F	T	T	F

Taking a look at the table, the three hypotheses are true in valuations (ie, rows) 1, 3, and 5, and in each of those three valuations C is also true, so the deduction is valid.

Exercise 2.52

1. (A v B) & (Y => Z), .: Y => Z

   &-elim with A = A v B and B = Y => Z

2. (A v B) & (Y => Z), .: (A v B) & (Y => Z)

   repeat with A = (A v B) & (Y => Z)

3. A v B, .: (A v B) v (Y => Z)

   v-intro with A = A v B and B = Y => Z

4. A v B, Y => Z, .: (A v B) & (Y => Z)

   &-intro with A = A v B and B = Y => Z

Exercise 2.53

(I'm doing extra work by specifying a translation key. The directions don't ask us to, but I'm just putting it here to make it easier for you to read.)

1. Proof by cases, with translation key:
   A = Susie will stop at the grocery store.
   B = Susie will stop at the drug store.
   C = Susie will buy milk.

2. &-intro, with translation key:
   A = My opponent is a liar.
   B = My opponent is a cheat.

3. Repeat, with translation key:
   A = John went to the store.

4. =>-elim, with translation key:
   A = I have $50.
   B = I am able to buy a new coat.

Chapter 3

Exercises 3.20, 3.25, 3.29, 3.30, and 3.33

Exercise 3.20

Give a two-column proof of the deduction
(P v Q) => (R & S), (R v S) => (P & Q), .: P => Q

```
1.  (P v Q) => (R & S)    Hypothesis
2.  (R v S) => (P & Q)    Hypotheses
3.  . P                   Assume
4.  . P v Q               v-intro line 3
5.  . R & S               =>-elim lines 1, 4
6.  . R                   &-elim line 5
7.  . R v S               v-intro line 6
8.  . P & Q               =>-elim lines 2, 7
9.  . Q                   &-elim line 8
10. P => Q                =>-intro lines 3-9
```

Exercises 3.25

Give a two-column proof for each of the deductions.

1

(P & -Q) => (Q v R), .: (P & -Q) => (R v S)

```
1. (P & -Q) => (Q v R)    Hypothesis
2. . P & -Q               Assume
3. . Q v R                =>-elim lines 1, 2
4. . -Q                   &-elim line 2
5. . R                    v-elim lines 3, 4
6. . R v S                v-intro line 5
7. (P & -Q) => (R v S)    =>-intro lines 2-6
```

2

P => (Q v R), Q => -R, R => S, .: P => S

```
1.  P => (Q v R)    Hypothesis
2.  Q => -P         Hypothesis
3.  R => S          Hypothesis
4.  . P             Assume
5.  . Q v R         =>-elim lines 1, 4
6.  . . Q           Assume
7.  . . -P          =>-elim lines 2, 6
8.  . . P & -P      &-intro lines 4, 7
9.  . -Q            Proof by contradiction lines 6-8
10. . R             v-elim lines 5, 9
```

```
11. . S              =>-elim lines 3, 10
12. P => S           =>-intro lines 4-11
```

Exercise 3.29

P: The Pope is here.
Q: The Queen is here.
R: The Russian is here.

(P & -Q) => R, .: P => (Q v R)

Proof by Contradiction

```
1. (P & -Q) => R        Hypothesis
2. . -(P => (Q v R))    Assume
3. . P & -(Q v R)       Log equiv line 2
4. . P & -Q & -R        Log equiv line 3
5. . P & -Q             &-elim line 4
6. . R                  =>-elim lines 1, 5
7. . -R                 &-elim line 4
8. . R & -R             &-intro lines 6, 7
9. P => (Q v R)         Proof by contradiction lines 2-8
```

Alternate Proof

```
1.  (P & -Q) => R      Hypothesis
2.  . P                Assume
3.  . Q v -Q           Tautology
4.  . . Q              Assume
5.  . . Q v R          v-intro line 4
6.  . Q => (Q v R)     =>-intro lines 4-5
7.  . . -Q             Assume
8.  . . P & -Q         &-intro lines 2, 7
9.  . . R              =>-elim lines 1, 8
10. . . Q v R          v-intro line 9
11. . -Q => (Q v R)    =>-intro lines 7-10
12. . Q v R            Proof by cases lines 3, 6, 11
13. P => (Q v R)       =>-intro lines 2-12
```

Exercises 3.30

Write a proof of each of these Theorems in English prose.

1

Hypotheses:
  1. If the Pope is here, then the Queen is here.
  2. If the Queen is here, then the Russian is here.

Conclusion:

If the Pope is here, then the Russian is here.

Proof:

Suppose the Pope is here. Since the Pope is here, Hypothesis 1 gives us that the Queen is here. Since the Queen is here, Hypothesis 2 gives us that the Russian is here. So, if the Pope is here, then so is the Russian.

2

THEOREM. Assume:

(a) If the Pope is here, then the Russian is here.
(b) If the Queen is here, then the Spaniard is here.
(c) The Pope and the Queen are both here.
Then the Russian and the Spaniard are both here.

Proof:

Hypothesis (c) gives us that the Pope is here, so Hypothesis (a) gives us that the Russian is here. Hypothesis (c) also gives us that the Queen is here, so Hypothesis (b) gives us that the Spaniard is here. Thus, both the Russian and the Spaniard are here.

3

THEOREM. Assume:

(a) If Adam is here, then Betty is here.
(b) If Betty is not here, then Charlie is here.
(c) Either Adam is here, or Charlie is not here.
Then Betty is here.

Proof:

We will consider two cases:
(Case 1) Assume Adam is here. Then by Hypothesis (a) Betty is here, so we're done.
(Case 2) Assume Adam is not here. Then Hypothesis (c) gives us that Charlie is not here. Since Charlie is not here, Hypothesis (b) tells us that Betty can't not be here (if she were not here, then Charlie would be here). To say Betty can't not be here is just to say that she is here, so we are done.

4

THEOREM. Assume:

(a) If Jack and Jill went up the hill, then something will go wrong.
(b) If Jack went up the hill, then Jill went up the hill.
(c) Nothing will go wrong.
Then Jack did not go up the hill.

Proof:
Assume for contradiction that Jack did go up the hill. Then hypothesis (b) gives us that Jill also went up the hill, so both Jack and Jill went up the hill. Then Hypothesis (a) gives us that something will go wrong. This contradict Hypothesis (c), so our assumption that Jack went up the hill must have been wrong, Jack must not have gone up the hill.

Exercise 3.33

1) Set A = T, B = F. Then A v B = T, but A => B = F.
2) Set P = T, Q = F. Then P v Q = T, but P & Q = F.
3) Set A = F, B = T, C = F. Then A => (B & C) = T, −A => (B v C) = T, but C = F.
4) Set P = F, Q = F, R = T. Then P => Q = T, −P => R = T, but Q & (P v R) = F.

Chapter 4

Exercises 4.9, 4.10, 4.24, 4.27, 4.31

Exercises 4.9

Provide a 2-column proof of each deduction.

1) (a ∈ A) ⇒ (a ∉ B), (b ∈ B) ⇒ (a ∈ B), ∴ (b ∈ B) ⇒ (a ∉ A)

```
1. (a ∈ A) ⇒ (a ∉ B)        Hypothesis
2. (b ∈ B) ⇒ (a ∈ B)        Hypothesis
3. . b ∈ B                  Assume (for ⇒-intro)
4. . a ∈ B                  ⇒-elim (lines 2, 3)
5. . ¬(a ∉ B)               Logical equivalence (line 4)
6. . ¬(a ∉ B) ⇒ ¬(a ∈ A)    Logical equivalence (line 1)
7. . ¬(a ∈ A)               ⇒-elim (lines 5, 6)
8. . a ∉ A                  Logical equivalence (line 7)
9. (b ∈ B) ⇒ (a ∉ A)        ⇒-intro (lines 3-8)
```

2) (p ∈ X) & (q ∈ X), (p ∈ X) ⇒ ((q ∉ X) v (Y = ∅)) , ∴ Y = ∅.

```
1. (p ∈ X) & (q ∈ X)                Hypothesis
2. (p ∈ X) ⇒ ((q ∉ X) v (Y = ∅))    Hypothesis
3. p ∈ X                            &-elim (line 1)
4. (q ∉ X) v (Y = ∅)                ⇒-elim (lines 2, 3)
5. q ∈ X                            &-elim (line 1)
6. Y = ∅                            v-elim (lines 4, 5)
```

A purist would require an intermediate step, where we write "¬(q ∈ X) v (Y = ∅)" with justification "Logical equiv. (line 4)". I won't require that level of detail fro you. We all understand that "q ∉ X" and "¬(q ∈ X)" mean exactly the same thing.

Exercises 4.10

Write your proofs in English.

1) Assume: (a) If p ∈ H and q ∈ H, then r ∈ H.
(b) q ∈ H.
Show that if p ∈ H, then r ∈ H.

Assume p ∈ H.
Then p ∈ H and q ∈ H.
Since p ∈ H and q ∈ H implies r ∈ H, we have r ∈ H.
Since assuming p ∈ H led to r ∈ H, we have show that p ∈ H implies r ∈ H.

(Note: Since we're trying to prove an "if ... then ..." statement, our first step should be to assume the antecedent. Then our proof is done once we have arrived at the consequent. [Look up those terms, "antecedent" and "consequent," in your book if you're forgotten them.])

(Other Note: A proof in Natural Language should be thought of as an outlines of a 2-column proof. You should write enough details so that one of your classmates can take your Natural Language proof and convert it into a 2-column proof without any significant work on their part. If you have to explain how to convert your English proof into a 2-column proof, then your English proof is not detailed enough.)

2) Assume: (a) If $X \neq \varnothing$, then $a \in Y$.
(b) If $X = \varnothing$, then $b \in Y$.
(c) If either $a \in Y$ or $b \in Y$, then $Y \neq \varnothing$.
Show $Y \neq \varnothing$.

We know that either $X \neq \varnothing$ or $X = \varnothing$, so we proceed in cases.
(case 1) Assume $X \neq \varnothing$.
Then $a \in Y$,
so either $a \in Y$ or $b \in Y$,
so $Y \neq \varnothing$ as desired.
(case 2) Assume $X = \varnothing$.
Then $b \in Y$,
so either $a \in Y$ or $b \in Y$,
so $Y \neq \varnothing$ as desired.
Since in either case we arrive at $Y \neq \varnothing$, we conclude that $Y \neq \varnothing$.

Exercises 4.24

Write a 2-column proof to justify each assertion.

1) $X \subset Y \Rightarrow X \subset Z$, $X \subset Z \Rightarrow x \in Z$, $x \notin Z$, $\therefore X \not\subset Y$.

(I'll give you a bonus. I'll do an English proof, and then convert it into a 2-col proof.)

We know $x \notin Z$. We want $X \not\subset Y$.
Since $x \notin Z$, by contrapositive, we get that $X \not\subset Z$.
Since $X \not\subset Z$, again contrapositive gives us $X \not\subset Y$, which is what we wanted.

1. $X \subset Y \Rightarrow X \subset Z$	Hypothesis
2. $X \subset Z \Rightarrow x \in Z$	Hypothesis
3. $x \notin Z$	Hypothesis
4. $x \notin Z \Rightarrow X \not\subset Z$	Logical equiv (line 1)
5. $X \not\subset Z$	$\Rightarrow$-elim (lines 3, 4)
6. $X \not\subset Z \Rightarrow X \not\subset Y$	Logical equiv (line 2)
7. $X \not\subset Y$	$\Rightarrow$-elim (lines 5, 6)

(Hopefully you can see what I mean about an English proof being an outline of a 2-col proof.)

2) (x ∈ Y) ⇒ (X ⊂ Y), (x ∈ Y) v (Y ⊂ X), ∴ (X ⊂ Y) v (Y ⊂ X).

```
1.  (x ∈ Y) ⇒ (X ⊂ Y)                  Hypothesis
2.  (x ∈ Y) v (Y ⊂ X)                  Hypothesis
3.  . x ∈ Y                            Assume (for ⇒-intro)
4.  . X ⊂ Y                            ⇒-elim (lines 1, 3)
5.  . (X ⊂ Y) v (Y ⊂ X)                v-intro (line 4)
6.  (x ∈ Y) ⇒ ((X ⊂ Y) v (Y ⊂ X))      ⇒-intro (lines 3-5)
7.  . (Y ⊂ X)                          Assume (for ⇒-intro)
8.  . (X ⊂ Y) v (Y ⊂ X)                v-intro (line 7)
9.  (Y ⊂ X) ⇒ ((X ⊂ Y) v (Y ⊂ X))      ⇒-intro (lines 7-8)
10. (X ⊂ Y) v (Y ⊂ X)                  Proof by cases (lines 2, 6, 9)
```

Exercises 4.27

Use the symbolization key on page 63, and write a 2-col proof for each.

1) (r ∈ S) ⇒ ((r O s) v (r ∉ S)), ∴ ((t ∈ S) & ¬(r O s)) ⇒ (r ∉ S)

```
1. (r ∈ S) ⇒ ((r O s) v (r ∉ S))     Hypothesis
2. . (t ∈ S) & ¬(r O s)              Assume (for ⇒-intro)
3. . . r ∈ S                         Assume (for contradiction)
4. . . (r O s) v (r ∉ S)             ⇒-elim (lines 1, 3)
5. . . ¬(r O s)                      &-elim (line 2)
6. . . r ∉ S                         v-elim (lines 4, 5)
7. . . (r ∈ S) & (r ∉ S)             &-intro (lines 3, 6)
8. . r ∉ S                           Proof by contradiction (lines
3-7)
9. ((t ∈ S) & ¬(r O s)) ⇒ (r ∉ S)    ⇒-intro (lines 2-8)
```

2) If either Roger is a student or Tess is not a student, then Sam is older than Tess. If Tess is a student, then Roger is also a student.
∴ Sam is older than Tess.

```
1.  ((r ∈ S) v (t ∉ S)) ⇒ (s O t)    Hypothesis
2.  (t ∈ S) ⇒ (r ∈ S)                Hypothesis
3.  . ¬((r ∈ S) v (t ∉ S))           Assume (for contradiction)
4.  . ¬(r ∈ S) & ¬(t ∉ S)            Logical equiv (line 3)
5.  . (r ∉ S) & (t ∈ S)              Logical equiv (line 4)
6.  . t ∈ S                          &-elim (line 5)
7.  . r ∈ S                          ⇒-elim (lines 2, 6)
8.  . r ∉ S                          &-elim (line 5)
9.  . (r ∈ S) & (r ∉ S)              &-intro (lines 7, 8)
10. (r ∈ S) v (t ∉ S)                Proof by contradiction (lines
```

3–9)
11. s O t ⇒–elim (lines 1, 10)

Exercises 4.31

Prove each deduction in English.

1) Assume A and B are sets, and let $C = \{a \in A \mid a \in B\}$.
Show that if $c \in C$, then $c \in B$.

We are trying to prove an implication, so assume the antecedent.
Assume $c \in C$. We need to show that $c \in B$.
By the definition of C, we have that C consists of all objects $a \in A$ such that $a \in B$, so we know that since $c \in C$, $c \in A$ and $c \in B$.
In particular, we have $c \in B$, as desired.

2) Let $A = \{x \in \mathbb{R} \mid x^2 - 5x = 14\}$.
Show that if $a \in A$, then $a < 10$.

Let $a \in A$.
By the definition of A, we have $a^2 - 5a = 14$.
By a little algebra, we have that either $a = 7$ or $a = -2$.
(case 1) If $a = 7$, then $a < 10$, as desired.
(case 2) If $a = -2$, then $a < 10$, as desired.
Since either way we get $a < 10$, we know that $a < 10$, as desired.

Chapter 5

Exercises 5.6, 5.17, 5.26

Exercises 5.6

1)

Suppose A and B are sets. Show that if $c \in A \cap B$, then $c \in A$.

Let $c \in A \cap B$.
Then $c \in A$ and $c \in B$.
In particular, $c \in A$.

2)

Suppose X, Y, and Z are sets.
Show that if $r \in (X \cap Y) \cup (X \cap Z)$, then $r \in X$.

Let $r \in (X \cap Y) \cup (X \cap Z)$.
Then $r \in X \cap Y$ or $r \in X \cap Z$.
(Case 1) Suppose $r \in X \cap Y$.
Then $r \in X$ and $r \in Y$.
In particular, $r \in X$.
(Case 2) Suppose $r \in X \cap Z$.
Then $r \in X$ and $r \in Z$.
In particular, $r \in X$.
In either case, we have $r \in X$ as desired.

Exercise 5.17

Suppose A and B are sets.
Show that if $c \in A' \cap B'$, then $c \in (A \cup B)'$.

Let $c \in A' \cap B'$.
Then $c \in A'$ and $c \in B'$.
That is, $c \notin A$ and $c \notin B$.
Symbolically, $\neg(c \in A)$ & $\neg(c \in B)$.
DeMorgan's gives $\neg(c \in A \vee c \in B)$.
This is equiv. to $\neg(c \in A \cup B)$,
which is to say $c \in (A \cup B)'$.

Exercises 5.26

1) Describe each of the following sets by listing its elements.

(a) $\mathcal{P}(\varnothing) = \{\varnothing\}$
(b) $\mathcal{P}(\{a\}) = \{\varnothing, \{a\}\}$
(c) $\mathcal{P}(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$
(d) $\mathcal{P}(\{a, b, c\})$

    = {∅, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}
(e) 𝒫({a, b, c, d})
    = {∅, {a}, {b}, {c}, {d},
       {a, b}, {a, c}, {a, d},
       {b, c}, {b, d}, {c, d},
       {a, b, c}, {a, b, d},
       {a, c, d}, {b, c, d},
       {a, b, c, d}}

2) Which are elements of 𝒫({a, b, c})?

(a) a ∉ 𝒫({a, b, c})
(b) {a} ∈ 𝒫({a, b, c})
(c) {a, b} ∈ 𝒫({a, b, c})

3) Suppose A is a set.

(a) Is ∅ ∈ 𝒫(A)? Why?

  Yes, because ∅ ⊂ A.

(b) Is A ∈ 𝒫(A)? Why?

  Yes, because A ⊂ A.

4) Does there exist a set A, such that 𝒫(A) = ∅?

  No matter what set A is, it's always the fact that ∅ ⊂ A, so we
always
  have at least ∅ ∈ 𝒫(A), so 𝒫(A) can never be empty.

5) Let V_0 = ∅
          V_1 = 𝒫(V_0)
          V_2 = 𝒫(V_1)
          and so forth.

(a) What are the cardinalities of V_0, V_1, V_2, V_3, V_4, and V_5?

  #(V_0) = 0
  #(V_1) = 1
  #(V_2) = 2
  #(V_3) = 4
  #(V_4) = 2^4 = 16
  #(V_5) = 2^16 = 65536

(b) List the elements of V_0, V_1, V_2, and V_3.

  V_0 = ∅
  V_1 = {∅}
  V_2 = {∅, {∅}}

```
  V_3 = {∅, {∅}, {{∅}}, {∅, {∅}}}

(c) List the elements of V_4.

  V_4 = {
    ∅,
    {∅},
    {{∅}},
    {{{∅}}},
    {{∅, {∅}}},
    {{∅}, {{∅}}},
    {{∅}, {{{∅}}}},
    {{∅}, {{∅, {∅}}}},
    {{{∅}}, {{{∅}}}},
    {{{∅}}, {{∅, {∅}}}},
    {{{{∅}}}, {{∅, {∅}}}},
    {∅, {∅}, {{∅}}},
    {∅, {∅}, {∅, {∅}}},
    {∅, {{∅}}, {∅, {∅}}},
    {{∅}, {{∅}}, {∅, {∅}}},
    {∅, {∅}, {{∅}}, {∅, {∅}}}
  }

(d) Is it reasonable to ask someone to list the elements of V_5? Why?

  Hell no! It's just too many elements. And it's really confusing (See
  V_4 above. How much more confusing would V_5 be?!). It'd actually
take
  more than a few minutes for a computer to work it out, even. This is
  the kind of thing that you need to describe by its characteristics
and
  features, because you can't really describe it by just saying what
it
  is.
```

Chapter 6

Exercises 6.10, 6.14, 6.15, 6.24

Exercises 6.10

Using the given symbolization key, translate each English-language assertion into First-Order Logic.

$\mathcal{U}$: The set of all people
D : The set of all ballet dancers.
F : The set of all females.
M : The set of all males.
x C y : x is a child of y.
x S y : x is a sibling of y.
e : Elmer
j : Jane
p : Patrick

1) Everyone who dances ballet is the child of someone who dances ballet.

$\forall x \in D, \exists y \in D, x \, C \, y$

2) Every man who dances ballet is the child of someone who dances ballet.

$\forall x \in D \cap M, \exists y \in D, x \, C \, y$

3) Everyone who dances ballet has a sister who also dances ballet.

$\forall x \in D, \exists y \in D \cap F, x \, S \, y$

4) Jane is an aunt.

$j \in F \mathbin{\&} \exists x, \exists y, x \, C \, y \mathbin{\&} y \, S \, j$

5) Patrick's brothers have no children.

$\forall x \in M, x \, S \, p \Rightarrow (\neg \exists y, y \, C \, x)$

Exercises 6.14

Negate each of the following assertions of First-Order Logic (and simplify, so that ¬ is not applied to anything but predicates or assertion variables.)

1) $(L \Rightarrow \neg M) \mathbin{\&} (M \lor N)$

$\neg ((L \Rightarrow \neg M) \mathbin{\&} (M \lor N))$

≡ ¬ (L ⇒ ¬ M) ∨ ¬ (M ∨ N)
≡ (¬ L & M) ∨ (¬ M & ¬ N)

2) ((a ∈ A) & (b ∈ B)) ∨ (c ∈ C)

¬ (((a ∈ A) & (b ∈ B)) ∨ (c ∈ C))
≡ ¬ ((a ∈ A) & (b ∈ B)) & ¬ (c ∈ C)
≡ (¬ (a ∈ A) ∨ ¬ (b ∈ B)) & ¬ (c ∈ C)
≡ (a ∉ A ∨ b ∉ B) & c ∉ C

3) ∀ a ∈ A, (((a ∈ P) ∨ (a ∈ Q)) & (a ∉ R))

¬ (∀ a ∈ A, (((a ∈ P) ∨ (a ∈ Q)) & (a ∉ R)))
≡ ∃ a ∈ A, ¬ (((a ∈ P) ∨ (a ∈ Q)) & (a ∉ R))
≡ ∃ a ∈ A, (¬ ((a ∈ P) ∨ (a ∈ Q)) ∨ ¬ (a ∉ R))
≡ ∃ a ∈ A, ((¬ (a ∈ P) & ¬ (a ∈ Q)) ∨ a ∈ R)
≡ ∃ a ∈ A, ((a ∉ P & a ∉ Q) ∨ a ∈ R)

4) ∀ a ∈ A, ((a ∈ T) ⇒ ∃ c ∈ C, ((c ∈ Q) & (c R a)))

¬ (∀ a ∈ A, ((a ∈ T) ⇒ ∃ c ∈ C, ((c ∈ Q) & (c R a))))
≡ ∃ a ∈ A, ¬ ((a ∈ T) ⇒ ∃ c ∈ C, ((c ∈ Q) & (c R a)))
≡ ∃ a ∈ A, ((a ∈ T) & ¬ ∃ c ∈ C, ((c ∈ Q) & (c R a)))
≡ ∃ a ∈ A, ((a ∈ T) & ∀ c ∈ C, ¬ ((c ∈ Q) & (c R a)))
≡ ∃ a ∈ A, ((a ∈ T) & ∀ c ∈ C, (¬ (c ∈ Q) ∨ ¬ (c R a)))
≡ ∃ a ∈ A, ((a ∈ T) & ∀ c ∈ C, (c ∉ Q ∨ ¬ (c R a)))

5) ∀ x, ((x ∈ A) & (∃ l ∈ L, ((x B l) ∨ (l ∈ C))))

¬ ∀ x, ((x ∈ A) & (∃ l ∈ L, ((x B l) ∨ (l ∈ C))))
≡ ∃ x, ¬ ((x ∈ A) & (∃ l ∈ L, ((x B l) ∨ (l ∈ C))))
≡ ∃ x, (¬ (x ∈ A) ∨ ¬ (∃ l ∈ L, ((x B l) ∨ (l ∈ C))))
≡ ∃ x, ((x ∉ A) ∨ (∀ l ∈ L, ¬ ((x B l) ∨ (l ∈ C))))
≡ ∃ x, ((x ∉ A) ∨ (∀ l ∈ L, (¬ (x B l) & (l ∉ C))))

6) A ⇒ ((∃ x ∈ X, (x ∈ B)) ∨ (∀ e ∈ E, ∃ d ∈ D, (e C d)))

¬ (A ⇒ ((∃ x ∈ X, (x ∈ B)) ∨ (∀ e ∈ E, ∃ d ∈ D, (e C d))))
≡ A & ¬ ((∃ x ∈ X, (x ∈ B)) ∨ (∀ e ∈ E, ∃ d ∈ D, (e C d)))
≡ A & (¬ (∃ x ∈ X, (x ∈ B)) & ¬ (∀ e ∈ E, ∃ d ∈ D, (e C d)))
≡ A & (∀ x ∈ X, x ∉ B) & (∃ e ∈ E, ∀ d ∈ D, ¬ (e C d))

7) ∀ a ∈ A, ∃ b ∈ B, ∃ c ∈ C, ∀ d ∈ D, (a K b) & ((a Z c) ∨ (b > d))

¬ ∀ a ∈ A, ∃ b ∈ B, ∃ c ∈ C, ∀ d ∈ D, (a K b) & ((a Z c) ∨ (b > d))
≡ ∃ a ∈ A, ∀ b ∈ B, ∀ c ∈ C, ∃ d ∈ D, ¬ ((a K b) & ((a Z c) ∨ (b > d)))
≡ ∃ a ∈ A, ∀ b ∈ B, ∀ c ∈ C, ∃ d ∈ D, ¬ (a K b) ∨ ¬ ((a Z c) ∨ (b > d))

≡ ∃ a ∈ A, ∀ b ∈ B, ∀ c ∈ C, ∃ d ∈ D, ¬ (a K b) ∨ (¬ (a Z c) & ¬ (b > d))

Exercises 6.15

Simplify each assertion. Show your work!

1) ¬ ∀ a ∈ A, (a ∈ P) ∨ (a ∈ Q)

¬ ∀ a ∈ A, (a ∈ P) ∨ (a ∈ Q)
≡ ∃ a ∈ A, ¬ ((a ∈ P) ∨ (a ∈ Q))
≡ ∃ a ∈ A, ¬ (a ∈ P) & ¬ (a ∈ Q)
≡ ∃ a ∈ A, a ∉ P & a ∉ Q

2) ¬ ∃ a ∈ A, (a ∈ P) & (a ∈ Q)

¬ ∃ a ∈ A, a ∈ P & a ∈ Q
≡ ∀ a ∈ A, ¬ (a ∈ P & a ∈ Q)
≡ ∀ a ∈ A, a ∉ P ∨ a ∉ Q

3) ¬ ∀ x ∈ X, ∃ y ∈ Y, ((x ∈ A) & (x C y))

¬ ∀ x ∈ X, ∃ y ∈ Y, ((x ∈ A) & (x C y))
≡ ∃ x ∈ X, ∀ y ∈ Y, ¬ ((x ∈ A) & (x C y))
≡ ∃ x ∈ X, ∀ y ∈ Y, x ∉ A ∨ ¬ (x C y)

4) ¬ ∀ s ∈ S, ((s ∈ R) ⇒ (∃ t ∈ T, ((s ≠ t) & (s M t))))

¬ ∀ s ∈ S, ((s ∈ R) ⇒ (∃ t ∈ T, ((s ≠ t) & (s M t))))
≡ ∃ x ∈ S, ¬ ((s ∈ R) ⇒ (∃ t ∈ T, ((s ≠ t) & (s M t))))
≡ ∃ x ∈ S, ((s ∈ R) & ¬ (∃ t ∈ T, ((s ≠ t) & (s M t))))
≡ ∃ x ∈ S, ((s ∈ R) & (∀ t ∈ T, ¬ ((s ≠ t) & (s M t))))
≡ ∃ x ∈ S, ((s ∈ R) & (∀ t ∈ T, (¬ (s ≠ t) ∨ ¬ (s M t))))
≡ ∃ x ∈ S, ((s ∈ R) & (∀ t ∈ T, ((s = t) ∨ ¬ (s M t))))

Exercises 6.24

Explain how you know that each of the following deductions is not valid.

1) ∃ x, (x ∈ A)
∃ x, (x ∈ B)
∴ ∃ x, ((x ∈ A) & (x ∈ B))

This deduction is invalid because we can easily think of a situation in
which the hypotheses are true but the conclusion is false. For instance,
let $\mathcal{U}$ = {1, 2}, let A = {1}, and let B = {2}. In this model, the hypotheses are both true, but the conclusion is false.

2) $\forall a \in A, \exists b \in B, (a \neq b)$

$A \neq \varnothing$

$\therefore \forall b \in B, \exists a \in A, (a \neq b)$

Let's think of a model in which the hypotheses are true but the conclusion is false. Let $\mathcal{U} = \{1, 2\}$, let $A = \{1\}$, and let $B = \{1, 2\}$. Then Hypothesis 1 is true because if $a = 1$, then $b = 2$, and that covers every element of A. Hypothesis 2 is true because $1 \in A$. But the conclusion fails because if we let $b = 1$, then there's no $a \in A$ such that $a \neq 1$.

3) $A \neq B$

$\therefore A \cup B \neq A$

Let's think. We want the conclusion to be false, so we want $A \cup B$ to be A. If B were a subset of A, then $A \cup B$ would be A, as desired. Let's draw up a formal counterexample: Let $\mathcal{U} = \{1, 2\}$. Let $A = \{1, 2\}$. Let $B = \{1\}$. Then $A \neq B$, but $A \cup B = A$, so the deduction is not valid.

4) $\forall x \in A, (x \notin B)$

$\forall x \in B, (x \notin A)$

$\therefore A \neq B$

This one is a little tricky. In words, Hypotheses 1 and 2 together say that the sets A and B must not have any elements in common. The conclusion then states that A and B must be different sets. The conclusion seems reasonable, but then we're forgetting something. Perhaps A and B are both the empty set. Then they would have no elements in common, yet they would both be the same set (i.e., the empty set). Let's make our counterexample: Let $\mathcal{U} = \{1\}$, let $A = B = \varnothing$. Then it is true (vacuously) that $\forall x \in A, x \notin B$, and similarly it is true (vacuously) that $\forall x \in B, x \notin B$, and yet we still have $A = B$, so the deduction is invalid.

Chapter 7

Exercises 7.21, 7.22, 7.23

Exercises 7.21

Suppose A and B are sets.

1) Show A \ B = A ∩ B′

Let x ∈ A \ B, so x ∈ A and x ∉ B.
Since x ∉ B, we have x ∈ B′.
Thus, since x ∈ A and x ∈ B′, we have x ∈ A ∩ B′.
This shows that ∀ x ∈ A \ B, x ∈ A ∩ B′,
which just means A \ B ⊂ A ∩ B′.

Now, let x ∈ A ∩ B′, so x ∈ A and x ∈ B′.
Since x ∈ B′, we have x ∉ B.
Thus, since x ∈ A and x ∉ B, we have x ∈ A \ B.
This shows that ∀ x ∈ A ∩ B′, x ∈ A \ B,
which just means A ∩ B′ ⊂ A \ B.

Since both A \ B ⊂ A ∩ B′ and A ∩ B′ ⊂ A \ B,
we must have that A \ B = A ∩ B′.

2) Show A = (A \ B) ∪ (A ∩ B)

(We will prove this equation with a slightly different method than what
was employed above. We will prove this equation by showing that the
predicate for being a member of the RHS (right-hand side) is logically
equivalent to the predicate "x ∈ A".)

Let x ∈ $\mathcal{U}$.

$$\begin{aligned} x \in (A \setminus B) \cup (A \cap B) &\equiv x \in (A \setminus B) \lor x \in (A \cap B) \\ &\equiv (x \in A \,\&\, x \notin B) \lor (x \in A \,\&\, x \in B) \\ &\equiv (x \in A) \,\&\, (x \notin B \lor x \in B) \\ &\equiv (x \in A) \,\&\, T \\ &\equiv x \in A \end{aligned}$$

Thus, ∀ x, x ∈ (A \ B) ∪ (A ∩ B) ⇔ x ∈ A,
so (A \ B) ∪ (A ∩ B) = A.

(Notes:
- We have (x ∉ B ∨ x ∈ B) ≡ T because (x ∉ B ∨ x ∈ B) is a tautology.
- We have (x ∈ A) & T ≡ x ∈ A because P & T ≡ P for any assertion

P.

- The algebra shows x ∈ (A \ B) ∪ (A ∩ B) ≡ x ∈ A, but then we go on to say x ∈ (A \ B) ∪ (A ∩ B) ⇔ x ∈ A. To do this, we have to recall one of the most important concepts from Propositional Logic: the idea that P ≡ Q means that P ⇔ Q is a tautology.
- ∀ x, x ∈ X ⇔ x ∈ Y says that the two sets X and Y have exactly the same elements, in which case, they must actually be the same set.
- We can, if we want to, solve this problem using the method we used on the first problem, i.e. show that the sets are subsets of each other. We took a different approach so that you could see examples of both methods.)

3) Prove De Morgan's Laws:

(a) (A′)′ = A

Let x ∈ 𝒰.

x ∈ (A′)′ ≡ x ∉ A′
≡ ¬ (x ∈ A′)
≡ ¬ (x ∉ A)
≡ ¬ (¬ x ∈ A)
≡ x ∈ A

Thus ∀ x, x ∈ (A′)′ ⇔ x ∈ A,
so we have (A′)′ = A.

(b) (A ∩ B)′ = A′ ∪ B′

(We'll use the method where we show that the LHS and RHS are subsets of each other.)

(w.t.s. LHS ⊂ RHS)
Let x ∈ (A ∩ B)′, so x ∉ A ∩ B,
so it is not the case that x ∈ A & x ∈ B,
which means that x ∉ A ∨ x ∉ B,
which is to say that x ∈ A′ ∨ x ∈ B′,
so x ∈ A′ ∪ B′.
This shows that ∀ x ∈ (A ∩ B)′, x ∈ A′ ∪ B′,
which is to say (A ∩ B)′ ⊂ A′ ∪ B′.

(w.t.s. RHS ⊂ LHS)
Let x ∈ A′ ∪ B′, so x ∈ A′ ∨ x ∈ B′,
so x ∉ A ∨ x ∉ B,

which means that it is not the case that x ∈ A & x ∈ B,
so it is not the case that x ∈ A ∩ B,
which is to say that x ∉ A ∩ B, that is x ∈ (A ∩ B)′.
This shows that ∀ x ∈ A′ ∪ B′, x ∈ (A ∩ B)′,
which is to say A′ ∪ B′ ⊂ (A ∩ B)′.

Finally, since the two sets are subsets of each other, they must actually be the same set, thus (A ∩ B)′ = A′ ∪ B′.

(c) (A ∪ B)′ = A′ ∩ B′

(We'll use the method where we show that membership in the LHS is logically equivalent to membership in the RHS.)

Let x ∈ 𝓤.

$$
\begin{aligned}
x \in (A \cup B)' &\equiv x \notin (A \cup B) \\
&\equiv \neg\,(x \in A \cup B) \\
&\equiv \neg\,(x \in A \lor x \in B) \\
&\equiv \neg\,(x \in A)\ \&\ \neg\,(x \in B) \\
&\equiv (x \notin A)\ \&\ (x \notin B) \\
&\equiv (x \in A')\ \&\ (x \in B') \\
&\equiv x \in A' \cap B'
\end{aligned}
$$

This shows that ∀ x, x ∈ (A ∪ B)′ ⇔ x ∈ A′ ∩ B′,
which is to say (A ∪ B)′ = A′ ∩ B′

4) Show that if A′ = B′, then A = B

(We want to prove an "if ... then ..." statement, so we assume the antecedent, and then it's our job to show the consequent.)

Assume A′ = B′.
We want to show that A = B.

(Now, we want to show that two sets are equal. We can do this by showing that they are subsets of each other. To do this, we'll at some point need to use the thing we assumed [i.e. A′ = B′], so try to watch for where it might be useful.)

(w.t.s. A ⊂ B)
Let x ∈ A.
We want to show that x ∈ B.
Since x ∈ A, we know x ∉ A′. (Here we used DM's(a): A′′ = A.)
Since A′ = B′, we know x ∉ B′.
Since x ∉ B′, we know x ∈ B.
Since letting x ∈ A resulted in having x ∈ B,
we have shown that ∀ x ∈ A, x ∈ B,
which is to say that A ⊂ B.

(w.t.s. $B \subset A$)
Let $x \in B$.
We want to show that $x \in A$.
Since $x \in B$, we know $x \notin B'$. (Again, this is where we used DM's(a).)
Since $A' = B'$, we know $x \notin A'$.
Since $x \notin A'$, we know $x \in A$.
Since letting $x \in B$ resulted in having $x \in A$,
we have shown that $\forall x \in B, x \in A$,
which is to say that $B \subset A$.

Since $A \subset B$ and $B \subset A$, we have $A = B$, as desired.

Exercises 7.22

Suppose A, B, and C are sets.

(Notes: to say that two sets X and Y are disjoint is the same as saying
$X \cap Y = \varnothing$. We will make use of this fact time and time again in these exercises.)

1) Show that A is disjoint from B if and only if $A \subset B'$

We need to show that $A \cap B = \varnothing \Leftrightarrow A \subset B'$.

(w.t.s. $A \cap B = \varnothing \Rightarrow A \subset B'$)
Let $A \cap B = \varnothing$.
We want to show that $A \subset B'$.
Let $x \in A$.
Since $A \cap B = \varnothing$, we know that there are no elements that are in both sets.
Symbolically, that is $\neg \exists x, x \in A \,\&\, x \in B$.
This is equivalent to $\forall x, \neg (x \in A \,\&\, x \in B)$,
which in turn is equiv. to $\forall x, x \notin A \vee x \notin B$,
which is equiv. to $\forall x, x \in A \Rightarrow x \notin B$.
Since $x \in A$, we can conclude that $x \notin B$.
Since $x \notin B$, we have $x \in B'$.
So far, we've shown that $\forall x \in A, x \in B'$,
which is to say $A \subset B'$.
Thus, $A \cap B = \varnothing \Rightarrow A \subset B'$.

(w.t.s. $A \subset B' \Rightarrow A \cap B = \varnothing$)
Let $A \subset B'$, so $\forall x \in A, x \in B'$.
We want to show that $A \cap B = \varnothing$.
Assume for contradiction that there is an $x \in \mathcal{U}$ that is in $A \cap B$.
Then $x \in A$ and $x \in B$.
Since $x \in A$ and $A \subset B'$, we'd have $x \in B'$, which means $x \notin B$.
But earlier we said $x \in B$, so this is a contradiction.
So, so far we have $\neg \exists x, x \in A \cap B$.
This is the same as $\forall x, x \notin A \cap B$,

so A ∩ B has no elements, so A ∩ B = ∅.
Thus, A ⊂ B′ ⇒ A ∩ B = ∅.

(Bring it all together)
Since A ∩ B = ∅ ⇒ A ⊂ B′ and A ⊂ B′ ⇒ A ∩ B = ∅,
we conclude that A ∩ B = ∅ ⇔ A ⊂ B′.

2) Show A \ B is disjoint from B

We want to show that (A \ B) ∩ B = ∅.
We will use proof by contradiction.

Assume for contradiction that (A \ B) ∩ B ≠ ∅.
Then, ∃ x, x ∈ (A \ B) ∩ B, so let's call it a.
We have a ∈ (A \ B) ∩ B, so a ∈ A \ B and a ∈ B.
Since a ∈ A \ B, we have a ∈ A and a ∉ B.
But earlier we said a ∈ B, so this is a contradiction,
so our assumption must be wrong.
Thus (A \ B) ∩ B = ∅.

3) Show that if A is disjoint from B, and C is a subset of B, then A is disjoint from C

(We want to show that if:
A ∩ B = ∅, and
C ⊂ B,
then
A ∩ C = ∅.
As always, we get to assume the hypotheses and it's our job to then
show the conclusion.)

Let A ∩ B = ∅ and let C ⊂ B.
(w.t.s. A ∩ C = ∅)
Since A ∩ B = ∅, we know ¬ ∃ x, x ∈ A ∩ B,
or in other words, ∀ x, x ∈ A ⇒ x ∉ B (simplification of above line).
Assume for contradiction that A ∩ C ≠ ∅, so ∃ x, x ∈ A ∩ C.
Let's take one such individual and call it a, so that a ∈ A ∩ C.
We have a ∈ A and a ∈ C.
Since a ∈ A and ∀ x, x ∈ A ⇒ x ∉ B, we get a ∉ B.
Since a ∈ C and C ⊂ B, we get a ∈ B.
This is a contradiction, so our assumption must be wrong.
Thus A ∩ C = ∅.

4) Show that A \ B is disjoint from A ∩ B

We want to show that (A \ B) ∩ (A ∩ B) = ∅.
We will use proof by contradiction.

Assume for contradiction that (A \ B) ∩ (A ∩ B) ≠ ∅.
Then ∃ x, x ∈ (A \ B) ∩ (A ∩ B).
Take one such individual and call it a, so a ∈ (A \ B) ∩ (A ∩ B).
We have a ∈ A \ B and a ∈ A ∩ B.
Since a ∈ A \ B, a ∈ A and a ∉ B, in particular a ∉ B.
Since a ∈ A ∩ B, a ∈ A and a ∈ B, in particular a ∈ B.
This is a contradiction, so our assumption must be wrong.
Thus (A \ B) ∩ (A ∩ B) = ∅.

5) Show that A is disjoint from B ∪ C iff A is disjoint from both B and C

We want to show A ∩ (B ∪ C) = ∅ ⇔ (A ∩ B = ∅ & A ∩ C = ∅).

(w.t.s. A ∩ (B ∪ C) = ∅ ⇒ (A ∩ B = ∅ & A ∩ C = ∅))
Let A ∩ (B ∪ C) = ∅.
(w.t.s A ∩ B = ∅ & A ∩ C = ∅)
Assume for contradiction ¬ (A ∩ B = ∅ & A ∩ C = ∅).
This simplifies to ¬ (A ∩ B = ∅) ∨ ¬ (A ∩ C = ∅),
which simplifies to (A ∩ B ≠ ∅) ∨ (A ∩ C ≠ ∅),
which translates to (∃ x, x ∈ A ∩ B) ∨ (∃ x, x ∈ A ∩ C).
We proceed in cases:
(case 1) Assume ∃ x, x ∈ A ∩ B.
Then we can pick one, call it a, so that a ∈ A ∩ B.
We have a ∈ A and a ∈ B.
Since a ∈ B, we have a ∈ B ∪ C.
Since a ∈ A and a ∈ B ∪ C, we have a ∈ A ∩ (B ∪ C),
So, ∃ x, x ∈ A ∩ (B ∪ C),
or in other words, A ∩ (B ∪ C) ≠ ∅,
which contradicts our initial premise.
(case 2) Assume ∃ x, x ∈ A ∩ C.
Then we can pick one, call it a, so that a ∈ A ∩ C.
We have a ∈ A and a ∈ C.
Since a ∈ C, we have a ∈ B ∪ C.
Since a ∈ A and a ∈ B ∪ C, we have a ∈ A ∩ (B ∪ C),
So, ∃ x, x ∈ A ∩ (B ∪ C),
or in other words, A ∩ (B ∪ C) ≠ ∅,
which contradicts our initial premise.
Since we get a contradiction in either case, we see that it is impossible for ¬ (A ∩ B = ∅ & A ∩ C = ∅) to be true,
thus we know A ∩ B = ∅ & A ∩ C = ∅.

(w.t.s. (A ∩ B = ∅ & A ∩ C = ∅) ⇒ A ∩ (B ∪ C) = ∅)
Let A ∩ B = ∅ & A ∩ C = ∅.
We translate this as (¬ ∃ x, x ∈ A ∩ B) & (¬ ∃ x, x ∈ A ∩ C),
which simplifies to (∀ x ∈ A, x ∉ B) & (∀ x ∈ A, x ∉ C).
(w.t.s. A ∩ (B ∪ C) = ∅)
(w.t.s. ¬ ∃ x, x ∈ A ∩ (B ∪ C))
(w.t.s. ∀ x ∈ A, x ∉ B ∪ C, we will use ∀-intro)
Let x ∈ A.

Then x ∉ B and x ∉ C.
This is equivalent to ¬ (x ∈ B ∨ x ∈ C),
which is equivalent to ¬ (x ∈ B ∪ C), or rather x ∉ B ∪ C.
Since we started with a general element of A,
we conclude that ∀ x ∈ A, x ∉ B ∪ C, as desired.

(Bring it all together)
Since both A ∩ (B ∪ C) = ∅ ⇒ (A ∩ B = ∅ & A ∩ C = ∅)
and (A ∩ B = ∅ & A ∩ C = ∅) ⇒ A ∩ (B ∪ C) = ∅,
we've shown A ∩ (B ∪ C) = ∅ ⇔ (A ∩ B = ∅ & A ∩ C = ∅).

Exercises 7.23

1) Show A ∪ B = (A \ B) ∪ (B \ A) ∪ (A ∩ B)

A ∪ B = {x | x ∈ A ∨ x ∈ B}.
(A \ B) ∪ (B \ A) ∪ (A ∩ B) = {x | x ∈ A \ B ∨ x ∈ B \ A ∨ x ∈ A ∩ B}.

Let x ∈ A ∪ B.
Then x ∈ A or x ∈ B.
(case 1) Assume x ∈ A.
We know x ∈ B or x ∉ B.
(case 1a) Assume x ∈ B.
Then x ∈ A ∩ B, so x ∈ (A \ B) ∪ (B \ A) ∪ (A ∩ B).
(case 1b) Assume x ∉ B.
Then x ∈ A \ B, so x ∈ (A \ B) ∪ (B \ A) ∪ (A ∩ B).
(case 2) Assume x ∈ B.
We know x ∈ A or x ∉ A.
(case 2a) Assume x ∈ A.
Then x ∈ A ∩ B, so x ∈ (A \ B) ∪ (B \ A) ∪ (A ∩ B).
(case 2b) Assume x ∉ A.
Then x ∈ B \ A, so x ∈ (A \ B) ∪ (B \ A) ∪ (A ∩ B).
Since x is a general element of A ∪ B,
we've shown that ∀ x ∈ A ∪ B, x ∈ (A \ B) ∪ (B \ A) ∪ (A ∩ B),
which is to say that A ∪ B ⊂ (A \ B) ∪ (B \ A) ∪ (A ∩ B).

Let x ∈ (A \ B) ∪ (B \ A) ∪ (A ∩ B).
Then x ∈ A \ B or x ∈ B \ A or x ∈ A ∩ B.
(case 1) Assume x ∈ A \ B.
Then x ∈ A and x ∉ B.
Since x ∈ A, we get x ∈ A ∪ B.
(case 2) Assume x ∈ B \ A.
Then x ∈ B and x ∉ A.
Since x ∈ B, we get x ∈ A ∪ B.
(case 3) Assume x ∈ A ∩ B.

Then $x \in A$ and $x \in B$.
In particular, $x \in A$, so $x \in A \cup B$.
Since x is a general element of $(A \setminus B) \cup (B \setminus A) \cup (A \cap B)$,
we've shown that $\forall\, x \in (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$, $x \in A \cup B$,
or in other words $(A \setminus B) \cup (B \setminus A) \cup (A \cap B) \subset A \cup B$.

Since both $A \cup B \subset (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$
and $(A \setminus B) \cup (B \setminus A) \cup (A \cap B) \subset A \cup B$,
we have that $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$.

2) Show the three sets $A \setminus B$, $B \setminus A$, and $A \cap B$ are all disjoint from each other

We need to show three things:
- $(A \setminus B) \cap (B \setminus A) = \varnothing$
- $(A \setminus B) \cap (A \cap B) = \varnothing$
- $(B \setminus A) \cap (A \cap B) = \varnothing$

Each of them we will prove by contradiction.

(w.t.s. $(A \setminus B) \cap (B \setminus A) = \varnothing$)
Assume for contradiction that $(A \setminus B) \cap (B \setminus A) \neq \varnothing$,
in other words, $\exists\, x$, $x \in (A \setminus B) \cap (B \setminus A)$.
We pick one such x and call it a,
so $a \in (A \setminus B) \cap (B \setminus A)$.
We have $a \in A \setminus B$ and $a \in B \setminus A$,
so $a \in A$ and $a \notin B$ and $a \in B$ and $a \notin A$.
Well, this is a contradiction if I've ever seen one,
so we conclude that $(A \setminus B) \cap (B \setminus A) = \varnothing$.

(w.t.s. $(A \setminus B) \cap (A \cap B) = \varnothing$)
Assume for contradiction that $(A \setminus B) \cap (A \cap B) \neq \varnothing$,
in other words, $\exists\, x$, $x \in (A \setminus B) \cap (A \cap B)$.
We pick on such x and call it a,
so $a \in (A \setminus B) \cap (A \cap B)$.
We have $a \in A \setminus B$ and $a \in A \cap B$,
so $a \in A$ and $a \notin B$ and $a \in A$ and $a \in B$.
We see that $a \notin B$ and $a \in B$, a contradiction,
so we conclude that $(A \setminus B) \cap (A \cap B) = \varnothing$.

(w.t.s. $(B \setminus A) \cap (A \cap B) = \varnothing$)
Assume for contradiction that $(B \setminus A) \cap (A \cap B) \neq \varnothing$,
in other words, $\exists\, x$, $x \in (B \setminus A) \cap (A \cap B)$.
We pick one such x and call it a,
so $a \in (B \setminus A) \cap (A \cap B)$.
We have $a \in B \setminus A$ and $a \in A \cap B$,
so $a \in B$ and $a \notin A$ and $a \in A$ and $a \in B$.
We see that $a \notin A$ and $a \in A$, a contradiction,
so we conclude that $(B \setminus A) \cap (A \cap B) = \varnothing$.

Chapter 9

Exercises 9.98, 9.100, 9.104, 9.109, 9.110, 9.111

Exercises 9.98

Let f : A → B and g : B → C.

1) Show that if f and g are bijections, then g ∘ f is a bijection.

Let f and g be bijections, so we know that
∀ b ∈ B, ∃! a ∈ A, f(a) = b and ∀ c ∈ C, ∃! b ∈ B, g(b) = c.
(w.t.s. ∀ c ∈ C, ∃! a ∈ A, (g ∘ f)(a) = c)
Let c ∈ C.
Since g is a bijection, there is a unique guy b, with b ∈ B and g(b) = c.
And, since f is a bijection, there is a unique guy a, with a ∈ A and f(a) = b.
We have that (g ∘ f)(a) = g(f(a)) = g(b) = c, so g ∘ f is at least onto.
We still need to show that a is unique (i.e., that g ∘ f is 1-to-1).
Assume that there is another guy a_2 ∈ A with (g ∘ f)(a_2) = c.
Then g(f(a_2)) = c = g(f(a)), so g(f(a_2)) = g(f(a)).
Since g is 1-to-1, we get f(a_2) = f(a),
and since f is 1-to-1, we get a_2 = a,
so g ∘ f is 1-to-1.
Since f is both onto and 1-to-1, f is a bijection.

2) Show that if g and g ∘ f are bijections, then f is a bijection.

Let g and g ∘ f be bijections, so we know that
∀ c ∈ C, ∃! b ∈ B, g(b) = c and ∀ c ∈ C, ∃! a ∈ A, (g ∘ f)(a) = c.
(w.t.s. ∀ b ∈ B, ∃! a ∈ A, f(a) = b)
Let b ∈ B. (w.t.s. ∃! a ∈ A, f(a) = b)
We have g(b) ∈ C.
Since g ∘ f is a bijection, we get a unique a ∈ A where (g ∘ f)(a) = g(b).
This is the same as saying g(f(a)) = g(b).
Since g is 1-to-1, we have f(a) = b, which shows that f is onto.
We still need to show that f is 1-to-1.
Assume that we have a_2 ∈ A where f(a_2) = b. (w.t.s. a_2 = a)
Assume that a_2 ≠ a. (w.t.f. a contradiction)
Since g ∘ f is a bijection, (g ∘ f)(a_2) ≠ (g ∘ f)(a),
which is the same as saying g(f(a_2)) ≠ g(f(a)).
Since g is a bijection, this gives f(a_2) ≠ f(a),
but earlier we had f(a) = b and f(a_2) = b, so b ≠ b, a contradiction.
Thus a_2 = a, so f is 1-to-1.
Since f is both onto and 1-to-1, f is a bijection.

3) Show that if f and g are bijections, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

We want to show that two functions are the same.
To do this, we can try to show that they do the same thing to points in C.
$(g \circ f)^{-1} : C \to A$.
$f^{-1} \circ g^{-1} : C \to A$.
Let $c \in C$. (w.t.s. $(g \circ f)^{-1}(c) = (f^{-1} \circ g^{-1})(c)$)
$(f^{-1} \circ g^{-1})(c) = f^{-1}(g^{-1}(c))$.
$g^{-1}(c)$ is the unique $b \in B$ with $g(b) = c$.
$f^{-1}(g^{-1}(c)) = f^{-1}(b)$, and $f^{-1}(b)$ is the unique $a \in A$ with $f(a) = b$.
$(g \circ f)^{-1}(c)$ is the unique $a_2 \in A$ with $(g \circ f)(a_2) = c$.
So we have $f^{-1}(g^{-1}(c)) = a$ and $(g \circ f)^{-1}(c) = a_2$. (w.t.s. $a_2 = a$)
Assume for contradiction that $a_2 \neq a$.
Take f of both sides. We get $f(a_2) \neq f(a)$ since f is a bijection.
Take g of both sides. We get $g(f(a_2)) \neq g(f(a))$ since g is a bijection.
Now, earlier we said $g(f(a_2)) = c$.
Also, we said $f(a) = b$ and that $g(b) = c$, so $g(f(a)) = c$.
But then this gives $c \neq c$, a contradiction, so $a_2 = a$.
So we have $\forall\, c \in C$, $(g \circ f)^{-1}(c) = (f^{-1} \circ g^{-1})(c)$,
so $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Exercises 9.100

1) Give an example of functions $f : A \to B$ and $g : B \to C$ such that $g \circ f$ is onto, but f is not onto.

Let $g : \mathbb{R} \to [0, \infty)$ by $g(x) = |x|$.
Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.
Then $g \circ f : \mathbb{R} \to [0, \infty)$.
$(g \circ f)(x) = |x^2| = x^2$, since x^2 is positive.
And $g \circ f$ is onto because x^2 hits every non-negative real.

2) Define $f : [0, \infty) \to \mathbb{R}$ by $f(x) = x$ and $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = |x|$. Show that $g \circ f$ is one-to-one, but g it not one-to-one.

Pt 1: (w.t.s. $g \circ f$ is one-to-one)
Let $x_1, x_2 \in [0, \infty)$. (w.t.s. $(g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow x_1 = x_2$)
Assume $x_1 \neq x_2$. (w.t.s. $(g \circ f)(x_1) \neq (g \circ f)(x_2)$)
$g(f(x_1)) = |x_1| = x_1$, since x_1 is non-negative.
$g(f(x_2)) = |x_2| = x_2$, since x_2 is non-negative.
So, $g(f(x_1)) \neq g(f(x_2))$.
Thus, $g \circ f$ is one-to-one.
Pt 2: (w.t.s. g is not one-to-one)
Consider $-1 \in \mathbb{R}$ and $1 \in \mathbb{R}$.
$g(-1) = 1 = g(1)$.
So, g is not one-to-one.

3) Suppose $f : A \to B$ and $g : B \to C$. Write a definition of $g \circ f$ purely in terms of sets of ordered pairs.

$$\begin{aligned}(a, c) \in g \circ f &\Leftrightarrow c = g(f(a)) \\ &\Leftrightarrow (f(a), c) \in g \\ &\Leftrightarrow \exists\, b \in B,\ b = f(a)\ \&\ (b, c) \in g \\ &\Leftrightarrow \exists\, b \in B,\ (a, b) \in f\ \&\ (b, c) \in g\end{aligned}$$

so $g \circ f = \{(a, c) \in A \times C \mid \exists\, b \in B,\ (a, b) \in f\ \&\ (b, c) \in g\}$

Exercises 9.104

Assume $f : A \to B$.
Let $A_1, A_2 \subset A$.

1) Show $A_2 \subset A_1 \Rightarrow f(A_2) \subset f(A_1)$

Assume $A_2 \subset A_1$. (w.t.s $f(A_2) \subset f(A_1)$)
Let $b \in f(A_2)$. (w.t.s. $b \in f(A_1)$)
Since $b \in f(A_2)$, there is an $a \in A_2$ where $f(a) = b$.
Since $A_2 \subset A_1$, $a \in A_1$.
Since $a \in A_1$, we have $f(a) \in f(A_1)$,
and $f(a) = b$, so $b \in f(A_1)$.
Thus $f(A_2) \subset f(A_1)$.

2) Assume f is 1-to-1 and $a \in A$. Show that if $f(a) \in f(A_1)$, then $a \in A_1$.

Assume $f(a) \in f(A_1)$. (w.t.s. $a \in A_1$)
Since $f(a) \in f(A_1)$, there is an $a_2 \in A_1$ where $f(a_2) = f(a)$.
Since f is 1-to-1, we get $a_2 = a$, so $a \in A_1$ as desired.

Exercises 9.109

Suppose that $f : A \to B$, that $A_1 \subset A$, and that $B_1 \subset B$.

1) Show that if $B_2 \subset B_1$, then $f^{-1}(B_2) \subset f^{-1}(B_1)$.

Assume $B_2 \subset B_1$. (w.t.s. $f^{-1}(B_2) \subset f^{-1}(B_1)$)
Let $a \in f^{-1}(B_2)$. (w.t.s. $a \in f^{-1}(B_1)$)
Since $a \in f^{-1}(B_2)$, we get $f(a) \in B_2$.
Since $B_2 \subset B_1$, we get $f(a) \in B_1$.
Since $f(a) \in B_1$, we get $a \in f^{-1}(B_1)$.
Thus, $f^{-1}(B_2) \subset f^{-1}(B_1)$.

2) Show $A_1 \subset f^{-1}(f(A_1))$.

Let $a \in A_1$. (w.t.s. $a \in f^{-1}(f(A_1))$)
Since $a \in A_1$, we get $f(a) \in f(A_1)$.
Since $f(a) \in f(A_1)$, we get $a \in f^{-1}(f(A_1))$, as desired.
Thus $A_1 \subset f^{-1}(f(A_1))$.

Exercises 9.110

Assume $f : X \to Y$, $A \subset Y$, and $B \subset Y$.
Show $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B)$.

Pt 1: (w.t.s. $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$)
Let $x \in f^{-1}(A) \cap f^{-1}(B)$. (w.t.s $x \in f^{-1}(A \cap B)$)
Then $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$.
Then $f(x) \in A$ and $f(x) \in B$.
Then $f(x) \in A \cap B$.
Then $x \in f^{-1}(A \cap B)$, as desired.
Thus $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$.
Pt 2: (w.t.s. $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$)
Let $x \in f^{-1}(A \cap B)$. (w.t.s. $x \in f^{-1}(A) \cap f^{-1}(B)$)
Then $f(x) \in A \cap B$.
Then $f(x) \in A$ and $f(x) \in B$.
Then $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$.
Then $x \in f^{-1}(A) \cap f^{-1}(B)$, as desired.
Thus $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$.
Thus $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B)$.

Exercises 9.111

Assume $f : X \to Y$, $g : Y \to Z$, $X_1 \subset X$, $Z_1 \subset Z$, and $(g \circ f)(X_1) \subset Z_1$.
Show $f(X_1) \subset g^{-1}(Z_1)$.

Let $y \in f(X_1)$. (w.t.s. $y \in g^{-1}(Z_1)$)
Since $y \in f(X_1)$, we have an $x \in X_1$ where $f(x) = y$.
Since $x \in X_1$, we have $(g \circ f)(x) \in (g \circ f)(X_1)$.
Since $(g \circ f)(X_1) \subset Z_1$, we have $(g \circ f)(x) \in Z_1$.
Since $(g \circ f)(x) = g(f(x))$, we have $g(f(x)) \in Z_1$.
Since $f(x) = y$, we have $g(y) \in Z_1$.
Since $g(y) \in Z_1$, we have $y \in g^{-1}(Z_1)$, as desired.

Chapter 10

Exercises 10.19, 10.27, 10.32, 10.47, 10.54

Exercises 10.19

Suppose A and B are finite sets, and $m, n \in \mathbb{N}$. Prove:

1) If $m \leq n$, then there exists a one-to-one function $f : \{1, 2, ..., m\} \to \{1, 2, ..., n\}$.

Let $m \leq n$.
(w.t.f. $f : \{1, 2, ..., m\} \to \{1, 2, ..., n\}$ s.t. f is 1-to-1)
Define $f : \{1, 2, ..., m\} \to \{1, 2, ..., n\}$ by $f(i) = i$.
We need to show that f is one-to-one.
(w.t.s. $\forall x_1, x_2 \in Dom(f), f(x_1) = f(x_2) \Rightarrow x_1 = x_2$)
Let $x_1, x_2 \in \{1, 2, ..., m\}$.
Assume $f(x_1) = f(x_2)$.
The formula for f says $f(x_1) = x_1$ and $f(x_2) = x_2$,
so $x_1 = x_2$.
Thus, f is one-to-one, as desired.

2) If $\#A \leq \#B$, then there exists a one-to-one function $f : A \to B$.

Let $\#A = m$. Let $\#B = n$. Then $m \leq n$.
Since $\#A = m$, there is a bijection $j : A \to \{1, ..., m\}$.
Since $\#B = n$, there is a bijection $k : B \to \{1, ..., n\}$.
From above problem, let $g : \{1, ..., m\} \to \{1, ..., n\}$ be 1-to-1.
Set $f = k^{-1} \circ g \circ j$.
Then $f : A \to B$, as desired,
and since f is a composition of 1-to-1 functions,
f is 1-to-1, as desired.

3) If $m \geq n$, then there exists an onto function $f : \{1, 2, ..., m\} \to \{1, 2, ..., n\}$.

Let $m \geq n$.
(w.t.f. $f : \{1, 2, ..., m\} \to \{1, 2, ..., n\}$ s.t. f is onto)

Define $f : \{1, 2, ..., m\} \to \{1, 2, ..., n\}$ piecewise by
$f(i) = \{ i$, if $i \leq n$
$\{ 1$, if $i > n$

We need to show that f is onto.
(w.t.s. $\forall y \in Codom(f), \exists x \in Dom(f), f(x) = y$)
Let $y \in \{1, 2, ..., n\}$.
(w.t.f. $x \in \{1, 2, ..., m\}$ s.t. $f(x) = y$)
Since $y \in \{1, 2, ..., n\}$, $y \leq n \leq m$, so $y \in \{1, 2, ..., m\}$.
Then, $f(y)$ makes sense (since $y \in \{1, 2, ..., m\}$) and $f(y) = y$.
Thus, f is onto.

4) If A and B are nonempty, and #A ≥ #B, then there exists an onto function f : A → B.

Let #A = m, so there is a bijection j : A → {1, ..., m}.
Let #B = n, so there is a bijection k : B → {1, ..., n}.
Since #A ≥ #B, we have m ≥ n.
Since m ≥ n, we get an onto function g : {1, ..., m} → {1, ..., n}.
Set $f = k^{-1} \circ g \circ j$.
Then f : A → B as desired,
and since f is a composition of onto functions,
f is onto, as desired.

Exercises 10.27

1) Show that #A is well-defined.

Say #A = m, so there is a bijection k : A → {1, ..., m}.
Say #A = n, so there is a bijection j : A → {1, ..., n}.
Then $k \circ j^{-1}$ is a bijection from {1, ..., n} to {1, ..., m},
so #{1, ..., n} = #{1, ..., m},
which would mean n = m.

2) Show ℕ is infinite.

(w.t.s. ¬ ∃ n ∈ ℕ, #ℕ = n)
Assume for contradiction that there is an n ∈ ℕ s.t. n = #ℕ.
Then there is a bijection f : ℕ → {1, ..., n}.
In particular, f is 1-to-1.
Now, consider the set {1, .., n, n+1} ⊂ ℕ.
Since f is 1-to-1, each x ∈ {1, ..., n, n+1}
must map to a different value in Codom(f).
But Codom(f) = {1, ..., n}.
By pigeonhole principle, there must be at least two
x ∈ {1, ..., n, n+1} that f maps to the same value.
This contradicts the earlier statement that f is 1-to-1,
so our original assumption must be false.
Thus there is no n ∈ ℕ s.t. n = #ℕ.

3) There are twelve people in a skating rink, playing ice hockey. Explain how you know that two of them were born on the same day of the week.

Pick seven of those 12 people. If two of those seven were born on the
same day of the week, then we win. If, on the other hand, none of
those seven were born on the same day of the week, then just pick one
more person from the remaining 5; since there are only seven days in
the week, the eighth person will share with one of the seven we

already picked, so we win either way.

4) If there are 700 students in a high school, explain how you know that there are two of them with the same initials.

There are 26 * 26 = 676 possible initials.
Since there are 700 students and only 676 possible initials,
there must be at least two students with the same initials.

5) Suppose A is a set of 10 natural numbers between 1 and 100 (inclusive). Show that two different subsets of A have the same sum.

There are 2^10 = 1024 subsets of a 10-element set.
The maximum possible sum is 100 + 99 + ... + 92 + 91 = 955,
and the minimum possible sum is 1 + 2 + ... + 9 + 10 = 55,
so there are 900 possible sums.
Since there 1024 subsets and only 900 possible sums,
there are at least two subsets that have the same sum.

Exercises 10.32

1) Suppose A and B are subsets of a finite set C. Show that if #A + #B > #C, then A ∩ B ≠ ∅.

Let #A + #B > #C.
Assume for contradiction A ∩ B = ∅.
Then #(A ∩ B) = 0.
Since A ⊂ C and B ⊂ C, we know A ∪ B ⊂ C,
so #(A ∪ B) ≤ #C.
Now, #C ≥ #(A ∪ B) = #A + #B − #(A ∩ B) = #A + #B.
Thus, #A + #B ≤ #C.
But this contradicts our hypothesis, so our assumption must be wrong.
Therefore, A ∩ B ≠ ∅.

2) Show that if A is a set of at least 600 natural numbers that are less than 1000, then two of the numbers in A differ by exactly 100.

Let B = {a + 100 | a ∈ A}.
Let C = {1, ..., 1100}.
Then A ⊂ C and B ⊂ C.
#A = 600, #B = 600, and #C = 1100,
so #A + #B > #C.
Thus, by the above problem, A ∩ B ≠ ∅.
Take x ∈ A ∩ B, so x ∈ A and x ∈ B.
Since x ∈ B, x = a + 100 for some a ∈ A.

Thus we have x, a ∈ A with x − a = 100.

Exercises 10.47

1) Suppose A is countably infinite, and b ∉ A. Show, directly from the definition, that A ∪ {b} is countably infinite.

Let A be countably infinite, so there is a bijection f : A → $\mathbb{N}^+$.

Define g : A ∪ {b} → $\mathbb{N}^+$ piecewise by
g(x) = { 1,   if x = b
       { f(x) + 1, if x ≠ b

We need to show g is a bijection.
(onto) Let y ∈ $\mathbb{N}^+$.
       If y = 1, then g(b) = 1.
       If y > 1, then since f is onto, y − 1 = f(x) for some x ∈ A,
       so y = f(x) + 1 = g(x).
       Thus, g is onto.
(1−to−1) Let x_1, x_2 ∈ A ∪ {b} with g(x_1) = g(x_2).
         If g(x_1) = g(x_2) = 1, then x_1 = x_2 = b.
         If g(x_1) = g(x_2) > 1, then g(x_1) − 1 = g(x_2) − 1.
         Now, g(x_1) − 1 = f(x_1), and g(x_2) − 1 = f(x_2),
         so the equation become f(x_1) = f(x_2),
         and since f is 1−to−1, we have x_1 = x_2.
         In either case, we get x_1 = x_2, so g is 1−to−1.
Thus, A ∪ {b} is countably infinite.

(In practice, you would not want to prove this directly from the definition. Chapter 10 gives some powerful theorems that can prove this result for us in two or three lines.)

2) Suppose A is countably infinite, and a ∈ A. Show, directly from the definition, that A \ {a} is countably infinite.

Since A is countably infinite, there is a bijection from A to $\mathbb{N}^+$.
Let f : A → $\mathbb{N}^+$ be one such bijection.

Remember, f is a set of ordered pairs.
Find the pair (b, f(b)) ∈ f.
Set g = {(x, f(x)) ∈ f | x ≠ b}
(we're basically deleting the pair (b, f(b)) from f),
so g : A \ {b} → $\mathbb{N}^+$,
and we know that g is 1−to−1 since f is 1−to−1,
but we also know that g is not onto, since g doesn't hit f(b).

Here's how we fix that.
Define g_2 : A \ {b} → $\mathbb{N}^+$ as follows:
If f(x) < f(b), then put (x, f(x)) ∈ g_2.
If f(x) ≥ f(b), then put (x, f(x) − 1) ∈ g_2.

Then, g_2 : A \ {b} → $\mathbb{N}^+$ is a bijection,
so A \ {b} is countably infinite.

(In practice, you would not want to prove this directly from the definition. Chapter 10 gives some powerful theorems that can prove this result for us in two or three lines.)

3) Suppose A and B are countably infinite and disjoint. Show, directly from the definition, that A ∪ B is countably infinite.

A is countably infinite, so there is a bijection f : A → $\mathbb{N}^+$.
B is countably infinite, so there is a bijection g : B → $\mathbb{N}^+$.

We need to find a bijection from A ∪ B to $\mathbb{N}^+$.

Define h : A ∪ B → $\mathbb{N}^+$ piecewise by
h(x) = { 2f(x),   if x ∈ A
       { 2g(x) + 1, if x ∈ B

(It's important to realize that we're using the fact that A and B are
disjoint when we define h, because we wouldn't be able to decide which
formula to use for an element x if it were both in A and in B.)

We need to show h is a bijection.

(1-to-1)
(w.t.s. ∀ x_1, x_2 ∈ Dom(h), h(x_1) = h(x_2) ⇒ x_1 = x_2)
Let x_1, x_2 ∈ A ∪ B.
Suppose h(x_1) = h(x_2).
For convenience, write n = h(x_1) = h(x_2).
n ∈ $\mathbb{N}^+$, so n is either even or odd.
(case 1) If n is even, then n = 2f(x_1) = 2f(x_2) by the def of h.
         Then f(x_1) = f(x_2),
         and since f is 1-to-1 we get x_1 = x_2.
(case 2) If n is odd, then n = 2g(x_1) + 1 = 2g(x_2) + 1 by def of h.
         Then g(x_1) = g(x_2),
         and since g is 1-to-1 we get x_1 = x_2.
In either case x_1 = x_2, so h is 1-to-1.

(onto)
(w.t.s. ∀ y ∈ Codom(h), ∃ x ∈ Dom(h), y = h(x))
Let y ∈ $\mathbb{N}^+$.
y is either even or odd.
(case 1) If y is even, then y = 2n for some n.
         Since f is onto, pick x ∈ A so that f(x) = n.
         Then y = 2f(x) = h(x), as desired.

(case 2) If y is odd, then y = 2n + 1 for some n.
Since g is onto, pick x ∈ B so that g(x) = n.
Then y = 2g(x) + 1 = h(x), as desired.
In either case, we find x ∈ A ∪ B so that h(x) = y, so h is onto.

Thus, h : A ∪ B → $\mathbb{N}^+$ is a bijection.
Therefore, A ∪ B is countably infinite.

4) Suppose
- A_1 is disjoint from B_1,
- A_1 and A_2 have the same cardinality,
- A_2 is disjoint from B_2, and
- B_1 and B_2 have the same cardinality.

Show that #(A_1 ∪ B_1) = #(A_2 ∪ B_2).

We have #A_1 = #A_2, #B_1 = #B_2, A_1 ∩ B_1 = ∅, and A_2 ∩ B_2 = ∅.

Since #A_1 = #A_2, there is a bijection f : A_1 → A_2.
Since #B_1 = #B_2, there is a bijection g : B_1 → B_2.

We want to show #(A_1 ∪ B_1) = #(A_2 ∪ B_2),
so we want to find a bijection h : A_1 ∪ B_1 → A_2 ∪ B_2.

Define h : A_1 ∪ B_1 → A_2 ∪ B_2 piecewise,
by h(x) = { f(x), if x ∈ A_1
{ g(x), if x ∈ B_2

We need to show that h is a bijection.

(1-to-1)
(w.t.s. ∀ x_1, x_2, ∈ Dom(h), h(x_1) = h(x_2) ⇒ x_1 = x_2)
Let x_1, x_2 ∈ A_1 ∪ B_1.
Assume h(x_1) = h(x_2).
Write y = h(x_1) = h(x_2).
y ∈ A_2 ∪ B_2, so y ∈ A_2 or y ∈ B_2.
(case 1) If y ∈ A_2,
then by the definition of h, y = f(x_1) = f(x_2).
Then, since f is 1-to-1, we have x_1 = x_2.
(case 2) If y ∈ B_2,
then by the definition of h, y = g(x_1) = g(x_2).
Then, since g is 1-to-1, we have x_1 = x_2.
In either case x_1 = x_2, so h is 1-to-1.

(onto)
(w.t.s. ∀ y ∈ Codom(h), ∃ x ∈ Dom(h), y = h(x))
Let y ∈ A_2 ∪ B_2, so y ∈ A_2 or y ∈ B_2.
(case 1) Suppose y ∈ A_2.
Since f is onto, there is an x ∈ A_1 such that y = f(x).
Then, by the definition of h, y = h(x) as well.
(case 2) Suppose y ∈ B_2.

Since g is onto, there is an $x \in B_1$ such that $y = g(x)$.
Then, by the definition of h, $y = h(x)$ as well.
In either case, we are able to find an $x \in A_1 \cup B_1$ with $y = h(x)$.
So h is onto.

Thus, $h : (A_1 \cup B_1) \to (A_2 \cup B_2)$ is a bijection.
Therefore $\#(A_1 \cup B_1) = \#(A_2 \cup B_2)$.

5) Suppose A is infinite. Show there is a proper subset B of A s.t. #B = #A.

Let A be infinite.

By theorem 10.41(1), A has a countably infinite subset, call it A_1.

Since A_1 is countably infinite, there is a bijection $f : A_1 \to \mathbb{N}^+$.

Let $A_2 = \{x \in A_1 \mid f(x) \text{ is even}\}$.

Since the even naturals are countably infinite (the function $2n$ is a bijection from $\mathbb{N}^+$ to the even naturals), we know that A_2 is also countably infinite, so $\#A_1 = \#A_2$.

Let $B = A \setminus A_1$,
so $A_1 \cap B = \varnothing$ and $A_1 \cup B = A$.

Since A_2 is a proper subset of A_1,
we have $A_2 \cap B = \varnothing$ and $A_2 \cup B$ is a proper subset of A.

Use 10.47(4), with $B = B_1 = B_2$, we get $\#(A_1 \cup B) = \#(A_2 \cup B)$,
and $A_1 \cup B = A$, so this says $\#(A) = \#(A_2 \cup B)$, as desired.

Exercise 10.54

Show that $\mathcal{P}(\mathbb{N}^+)$ is uncountable.

Assume that $\mathcal{P}(\mathbb{N}^+)$ is countable,
so there is a bijection $f : \mathbb{N}^+ \to \mathcal{P}(\mathbb{N}^+)$.
Let $A = \{i \in \mathbb{N}^+ \mid i \notin f(i)\}$, so $A \subset \mathbb{N}^+$.
f is onto, so there must be a $k \in \mathbb{N}^+$ s.t. $f(k) = A$.
Now, either $k \in A$ or $k \notin A$.
If $k \in A$, then $k \notin f(k)$, but $A = f(k)$, so this is impossible.
If $k \notin A$, then $k \in f(k)$, but $A = f(k)$, so this is impossible.
But if both can't be impossible, so we have a contradiction.
Thus, our original assumption, that $\mathcal{P}(\mathbb{N}^+)$ is countable, is wrong.
Therefore, $\mathcal{P}(\mathbb{N}^+)$ is uncountable.

Chapter 11

Exercises 11.14 (odd), 11.25, 11.38 (2), 11.50 (2), 11.52

Exercises 11.14 (odd)

1) sum_(k=1)^n (6k + 7) = 3n^2 + 10n

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  (w.t.s. ∀ n ∈ ℕ⁺, sum_(k=1)^n (6k + 7) = 3n^2 + 10n)
  (base case)
    sum_(k=1)^1 (6k + 7) = 6 + 7 = 13
    3(1)^2 + 10(1) = 3 + 10 = 13
    so sum_(k=1)^1 (6k + 7) = 3(1)^2 + 10(1)
  (induction step)
    Assume sum_(k=1)^j (6k + 7) = 3j^2 + 10j
    (w.t.s. sum_(k=1)^(j+1) (6k + 7) = 3(j+1)^2 + 10(j+1))
    (left side) sum_(k=1)^(j+1) (6k + 7)
                = sum_(k=1)^j (6k + 7) + 6(j+1) + 7
                = (3j^2 + 10j) + 6(j+1) + 7
                = 3j^2 + 16j + 13
    (right side) 3(j+1)^2 + 10(j+1)
                 = 3(j^2 + 2j + 1) + 10j + 10
                 = 3j^2 + 6j + 3 + 10j + 10
                 = 3j^2 + 16j + 13
    so sum_(k=1)^(j+1) (6k + 7) = 3j^2 + 16j + 13
  So, but PMI, ∀ n ∈ ℕ⁺, sum_(k=1)^n (6k + 7) = 3n^2 + 10n
```

3, 5, 7 are exactly the same.

Exercises 11.25

1) $\forall n \in \mathbb{N}^+$, 3^n ≥ 3n

```
  (base case)
    (w.t.s. 3^1 ≥ 3(1))
    3^1 = 3
    3(1) = 3
    so 3^1 ≥ 3(1)
  (induction step)
    Assume 3^k ≥ 3k for some number k
    (w.t.s. 3^(k+1) ≥ 3(k+1))
    3^(k+1) = 3(3^k) ≥ 3(3k) = 9k ≥ 3k + 3 = 3(k+1)
  So by induction, ∀ n ∈ ℕ⁺, 3^n ≥ 3n.
```

2) $\forall x \in \mathbb{R}^+$, $\forall n \in \mathbb{N}^+$, (1 + x)^n ≥ 1

```
  Let x ∈ ℝ⁺
  (w.t.s. ∀ n ∈ ℕ⁺, (1 + x)^n ≥ 1)
  (base case)
    (w.t.s. (1 + x)^1 ≥ 1)
```

```
    Since x ∈ ℝ⁺, 1 + x > 1,
    so (1 + x)^1 ≥ 1.
  (induction step)
    Assume (1 + x)^k ≥ 1 for some k.
    (w.t.s. (1 + x)^(k+1) ≥ 1)
    (1 + x)^(k+1)
    = (1 + x)(1 + x)^k
    and since (1 + x)^k ≥ 1
    we get (1 + x)(1 + x)^k ≥ (1 + x)(1) = (1 + x)
    so (1 + x)^(k+1) ≥ 1 + x
    and since x ∈ ℝ⁺, we have 1 + x ≥ 1
    so (1 + x)^(k+1) ≥ 1 + x ≥ 1
    or simply (1 + x)^(k+1) ≥ 1
    as desired.
  By induction, we've shown ∀ n ∈ ℕ⁺, (1 + x)^n ≥ 1.
  Since x was arbitrary in ℝ⁺,
  we've shown ∀ x ∈ ℝ⁺, ∀ n ∈ ℕ⁺, (1 + x)^n ≥ 1.
```

Exercise 11.38 (2)

```
Prove Theorem 11.33 (Every nonempty subset of ℕ has a smallest
element)

  Let P(n) : If S ⊂ ℕ and ∃ x ∈ S, x ≤ n, then S has a smallest
element.

  (base case)
    (w.t.s. P(1))
    (w.t.s. If S ⊂ ℕ and ∃ x ∈ S, x ≤ 1, then S has a smallest
element)
    Let S ⊂ ℕ and let x ∈ S with x ≤ 1.
    (w.t.s. S has a smallest element)
    Either x < 1 or x = 1
    (case 1) Assume x < 1.
             (w.t.s. S has a smallest element)
             Then x = 0,
             and no natural number is smaller than 0,
             so S has a smallest element, namely 0.
    (case 2) Assume x = 1.
             (w.t.s. S has a smallest element)
             Either 0 ∈ S or 0 ∉ S.
             (case 1) Assume 0 ∈ S.
                      (w.t.s. S has a smallest element)
                      Then 0 has to be the smallest element,
                      so S has a smallest element.
             (case 2) Assume 0 ∉ S.
                      (w.t.s. S has a smallest element)
                      We have 0 ∉ S and 1 ∈ S,
                      so 1 is the smallest element.
                      Thus, S has a smallest element.
```

```
            So S has a smallest element,
            since it has a smallest element in either case.
    So overall, we know that S must have a smallest element,
    since it has a smallest element in either case.
  (induction step)
    Assume that If S ⊂ ℕ⁺ and ∃ x ∈ S, x ≤ k, then S has a smallest
element.
    (w.t.s. P(k+1))
    Let S ⊂ ℕ⁺ and ∃ x ∈ S, x ≤ k+1.
    (w.t.s. S has a smallest element)
    Either x < k + 1 or x = k + 1.
    (case 1) Assume x < k + 1.
             (w.t.s. S has a smallest element)
             Since x < k + 1, x ≤ k.
             Then by the induction hypothesis, S has a smallest
element.
    (case 2) Assume x = k + 1.
             (w.t.s. S has a smallest element)
             Either ∃ y ∈ S, y < x or ¬ ∃ y ∈ S, y < x.
             (case 1) Assume ∃ y ∈ S, y < x.
                      Since x = k + 1 and y < x, we have y ≤ k,
                      so by the induction hypothesis, S has a smallest
element.
             (case 2) Assume ¬ ∃ y ∈ S, y < x.
                      Then there's nothing in S that is smaller than
x,
                      so x is the smallest element of S,
                      so S has a smallest element.
    Thus, S has a smallest element.
  By the Principle of Mathematical Induction,
  we've shown ∀ n ∈ ℕ⁺, (S ⊂ ℕ & (∃ x ∈ S, x ≤ n)) ⇒ S has a smallest
element.

  We still need to prove the Well-Ordering Priciple.
  (w.t.s. ∀ X, (X ⊂ ℕ & X ≠ ∅) ⇒ X has a smallest element)
  Let X ⊂ ℕ and assume X ≠ ∅.
  (w.t.s. X has a smallest element)
  Since X ≠ ∅, there is at least one element in X, say k ∈ X.
  So, X ⊂ ℕ & (∃ x ∈ S, x ≤ k) is true of the set X, since k ∈ X.
  Well, since X ⊂ ℕ & (∃ x ∈ S, x ≤ k) is true,
  and because "∀ n ∈ ℕ⁺, (S ⊂ ℕ & (∃ x ∈ S, x ≤ n)) ⇒ S has a smallest
element"
  is true (that's what we have shown above)
  we have that X has a smallest element.
```

Exercise 11.50 (2)

Suppose the sets A_1, A_2, ..., A_n are pairwise-disjoint.
Show A_n is disjoint from A_1 ∪ A_2 ∪ ... ∪ A_(n - 1) if n > 1.

(we need to use generalize induction, because our base case is P(2))
(base case)
(w.t.s. A_2 is disjoint from A_1)
A_2 is disjoint from A_1
since all the A_i are pairwise-disjoint.
(induction step)
Assume A_k is disjoint from A_1 ∪ ... ∪ A_(k−1) for some k ≥ 2.
(w.t.s. A_(k+1) is disjoint from A_1 ∪ ... ∪ A_k)
A_1 ∪ ... ∪ A_k = (A_1 ∪ ... ∪ A_(k−1)) ∪ A_k.
By the induction hypothesis, A_(k+1) is disjoint from A_1 ∪ ... ∪ A_(k−1),
and A_(k+1) is disjoint from A_k since the A_i are pairwise-disjoint,
so A_(k+1) is disjoint from their union,
that is, A_(k+1) is disjoint from A_1 ∪ ... ∪ A_k, as desired.
So for all n > 1, A_n is disjoint from A_1 ∪ A_2 ∪ ... ∪ A_(n − 1).

11.52

The induction step of the proof requires that we have at least three horses in the set H, because it says we need h_1, h_2, and h to all be different horses. This works if n ≥ 3, but it doesn't work for n = 2.

(Think of induction as dominos. We have the base case of P(1), and the induction step shows that P(3) ⇒ P(4) ⇒ P(5) ⇒ ..., but we're still missing P(2), so no chain reaction.)